A Probabilistic Two-Pile Game

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June 5-7, 2019

Introduction

Game is the source of motivation to do mathematics. A will to win a game is the motivation to figure out mathematics behind it. It is evident that pile games have been played since ancient times. For examples, Nim, Wythoff and their variants are some of the classical pile games.

In Nim, a game with multiple piles, each player may remove any number of chips from one of the available piles. The player who takes the last chip loses the game (misere game). In some other games, the player who takes the last chip wins the game (normal play).

Introduction

Recently, Wong and Xu [5] studied a game where two players take turns to collect a specific number of chips from the set S randomly and independently in order to build their own piles (instead of ‘remove any chip from their own piles’). The player who collects n chips first is the winner.

Definition of p_n

Let the random variable S_k be the number of chips collected by a player on his k^{th} move. Let A be the first player and B be the second player. The probability that B wins the game by collecting n chips first is

$$p_n = \sum_{k=1}^{\infty} P(\text{A does not win on his } k^{th} \text{ move}) \cdot P(\text{B wins on his } k^{th} \text{ move}). \tag{1}$$

Important Definitions

- p_n = the probability for the second player to win the game by collecting n chips first.
- $q(n,k)$ = the probability that a player does not win the game on his k^{th} move, i.e., he never collects n chips on or before his k^{th} move.
- $r(n,k)$ = the probability that a player collects n chips for the first time on his k^{th} move.

The equation (1) can be written as follows:

$$p_n = \sum_{k=1}^{\infty} q(n,k) \cdot r(n,k) \qquad (2)$$

where $q(n,k) = P(S_j < n \text{ for all } j = 0, 1, \ldots, k)$ and $r(n,k) = P(S_k \geq n \text{ and } S_j < n \text{ for all } j = 0, 1, \ldots, k-1)$.

The Claims

Claim 1: For any positive integer n,

$$r(n,k) = q(n,k-1) - q(n,k), \qquad k = 1, 2, \ldots.$$

Also

$$q(n,k) = 1 - \sum_{j=1}^{k} r(n,j). \tag{3}$$

Claim 2: Let n be a fixed positive integer. If $\lim_{k\to\infty} q(n,k) = 0$, then

$$\sum_{k=1}^{\infty} \big(q(n,k-1) + q(n,k)\big) \cdot r(n,k) = 1.$$

Useful Theorem

Theorem (1)

If $\lim_{k\to\infty} q(n,k) = 0$, *then* $p_n = \frac{1}{2} - \frac{1}{2}\sum_{k=1}^{\infty} r^2(n,k)$.

Proof.

By combining the two equations. The first one by definition and the second one by claim 2 and claim 1.

$$p_n = \sum_{k=1}^{\infty} q(n,k) \cdot r(n,k).$$

$$p_n = 1 - \sum_{k=1}^{\infty} q(n,k-1) \cdot r(n,k).$$

□

Previous Work

For the case $\{a, b\} = \{1, 2\}$ (Each turn adding 1 chip or 2 chips to the pile randomly), Wong and Xu [5, Theorem 3] obtained the following expression for the probability $r(n, k)$:

$$r(n,k) = \frac{1}{2^k}\left(\binom{k}{n-k} + \binom{k-1}{n-k}\right).$$

In particular, they showed that

$$\sum_{k=1}^{\infty} r^2(n,k) \sim \sqrt{\frac{27}{8\pi n}}$$

when n is large.

The case $\{a, b\} = \{-1, 1\}$

Each player adds/removes one chip with probability 1/2 to/from his pile. The pile is allowed to have a negative number of chips. The first player who collects n chips wins the game.

The winning probability for the first non-trivial case: $n = 1$

Let $C(k)$ be the number of ways for a player to have no chip on his k^{th} move without ever collecting one chip (so the game still goes on).
We have

$$C(2m-1) = 0$$

and

$$C(2m) = \frac{\binom{2m}{m}}{m+1}.$$

The winning probability for the first non-trivial case: $n = 1$

The probability that the second player collects one chip for the first time on his k^{th} move is

$$r(k) = \frac{C(k-1)}{2^k}$$

Hence, for $m = 1, 2, \ldots,$ we have

$$r(2m) = 0$$

and

$$r(2m-1) = \frac{(2m-2)!}{m!(m-1)!} \cdot \frac{2}{4^m}.$$

The winning probability for the first non-trivial case: $n = 1$

We show that

$$\sum_{k=1}^{\infty} r(k) = 1. \tag{4}$$

For example, in Maple, one conveniently types

```
sum( (2*m-2)!/m!/(m-1)!*2/4^m, m=1..M);
```

then Maple will return the expression $1 - \frac{(2M)!}{M!M!4^M}$. By an application of the Stirling formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we note that

$$1 - \frac{(2M)!}{M!M!4^M} \to 1 \text{ as } M \to \infty.$$

Hence, the equation (4) is true.

The winning probability for the first non-trivial case: $n = 1$

To evaluate

$$\sum_{k=1}^{\infty} r^2(k),$$

in Maple, we type

```
sum( ( (2*m-2)!/m!/(m-1)!*2/4^m )^2, m=1..infinity);
```

and then Maple will return $\frac{4}{\pi} - 1$. By Theorem 1,

$$p_1 = \frac{1}{2} - \frac{1}{2}\left(\frac{4}{\pi} - 1\right) \approx 0.3633802277.$$

The winning probabilities for the cases $n \geq 2$

Analogously, let $C(n,k)$ be the number of ways for a player to have $n-1$ chips on his k^{th} move without ever collecting n chips.
The main recurrence relation for the numbers $C(n,k)$ is

$$C(n,k) = C(n-1,k-1) + C(n+1,k-1), \quad n \geq 1, \quad k \geq 1.$$

We rearrange terms and shift variables to obtain

$$C(n,k) = C(n-1,k+1) - C(n-2,k), \quad n \geq 2, \quad k \geq 0. \tag{5}$$

The winning probabilities for the cases $n \geq 2$

Lemma

For $n \geq 0, \quad k \geq 1$ *such that* $(n-k) \equiv 0 \mod 2$, *then*

$$C(n,k) = 0.$$

Otherwise, for $s \geq 0$,

$$C(2s+1, 2m) = \frac{2s+1}{m+s+1}\binom{2m}{m-s}, \quad \text{for } m = 0, 1, 2, \ldots,$$

$$C(2s, 2m-1) = \frac{s}{m}\binom{2m}{m-s}, \quad \text{for } m = 1, 2, \ldots, .$$

The winning probabilities for the cases $n \geq 2$

$$r(n,k) = \frac{C(n,k-1)}{2^k}. \tag{6}$$

By Lemma 2, for $n \geq 0$, $k \geq 1$ and $n - k \equiv 1 \mod 2$, we have

$$r(n,k) = 0. \tag{7}$$

Otherwise, for $s \geq 0$,

$$r(2s+1, 2m+1) = \frac{C(2s+1, 2m)}{2^{2m+1}} = \frac{2s+1}{(m+s+1)\cdot 2 \cdot 4^m}\binom{2m}{m-s},$$

$$\text{for } m = 0, 1, 2, \ldots,$$

$$r(2s, 2m) = \frac{C(2s, 2m-1)}{2^{2m}} = \frac{s}{m \cdot 4^m}\binom{2m}{m-s},$$

$$\text{for } m = 1, 2, \ldots.$$

The winning probabilities for the cases $n \geq 2$

Lemma

For each $n \geq 1$, the probability that one of the players wins the game (by collecting n chips first) is 1, i.e.,

$$\sum_{k=1}^{\infty} r(n,k) = 1.$$

The proof can be done by induction on n. The case $n = 1$ has already been derived.

The winning probabilities for the cases $n \geq 2$

Finally, we are in the position to apply Theorem 1 for each value of n. For example, to evaluate $\sum_{k=1}^{\infty} r^2(2,k)$, we type

```
sum( (binomial(2*m,m-1)/m/4^m)^2 ,m=1..infinity);
```

and then Maple will return $\frac{16}{\pi} - 5$. By Theorem 1,

$$p_2 = \frac{1}{2} - \frac{1}{2}\left(\frac{16}{\pi} - 5\right) \approx 0.4535209109.$$

The winning probabilities for the cases $n \geq 2$

The other values of p_n are

$$n = 3, \quad \sum_{k=1}^{\infty} r^2(3,k) = \frac{236}{3\pi} - 25 \text{ and } p_3 \approx 0.4798111434,$$

$$n = 4, \quad \sum_{k=1}^{\infty} r^2(4,k) = \frac{1216}{3\pi} - 129 \text{ and } p_4 \approx 0.4891964033,$$

$$n = 5, \quad \sum_{k=1}^{\infty} r^2(5,k) = \frac{32092}{15\pi} - 681 \text{ and } p_5 \approx 0.4933044576,$$

$$n = 6, \quad \sum_{k=1}^{\infty} r^2(6,k) = \frac{172144}{15\pi} - 3653 \text{ and } p_6 \approx 0.4954322531.$$

. . . .

Pattern for p_n

There is an interesting pattern for the values of $\sum_{k=1}^{\infty} r^2(n,k)$ for $n = 1, 2, \ldots$. In fact, let T_n be $\sum_{k=1}^{\infty} r^2(n,k)$. The terms T_n appear to satisfy a recurrence relation with polynomial coefficients:

$$(n+3)T_{n+3} - (7n+16)T_{n+2} + (7n+5)T_{n+1} - nT_n = 0. \qquad (8)$$

Pattern for p_n

In this case, the asymptotic approximation of T_n can be obtained, (refer to the beautiful article by J. Wimp and D. Zeilberger, [4]) :

$$T_n = \frac{1}{\pi n^2}\left(1 + \frac{1}{n^2} + \frac{19}{4n^4} + \frac{107}{2n^6} + \dots\right) + O(\alpha^n),$$

for some α such that $|\alpha| < 1$.

The winning probability within k moves, $n = 1$

The case $\{a, b\} = \{-1, 1\}$ is so nice that we can even answer more questions than the one that was asked in the first place.
We demonstrate it for the case $n = 1$. The other case of n can be done similarly. We recall from previously that

$$r(2m) = 0$$

and

$$r(2m-1) = \frac{(2m-2)!}{m!(m-1)!} \cdot \frac{2}{4^m}.$$

The winning probability within k moves, $n = 1$

By the remark after Claim 1,

$$q(2m) = q(2m-1) = 1 - \sum_{j=1}^{m} r(2j-1) = 1 - \sum_{j=1}^{m} \frac{(2j-2)!}{j!(j-1)!} \cdot \frac{2}{4^j}.$$

This finite sum is Gosper-summable. In Maple, we type

```
simplify( 1-sum( (2*j-2)!*2/j!/(j-1)!/4^j ,j=1..m));
```

and then the output is $\frac{\binom{2m}{m}}{4^m}$.

The winning probability within k moves, $n = 1$

Lastly, the probability that the second player wins the game within k moves (the partial sum of (2)) is

$$\sum_{i=1}^{k} q(i)r(i) = \sum_{j=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} q(2j-1)r(2j-1)$$

$$= \sum_{j=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \frac{\binom{2j}{j}}{4^j} \frac{(2j-2)!}{j!(j-1)!} \cdot \frac{2}{4^j}$$

$$= \sum_{j=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \binom{2j}{j}^2 \cdot \frac{1}{16^j} \cdot \frac{1}{2j-1}.$$

The winning probability within k moves, $n = 1$

In Maple, we type

```
sum(binomial(2*j,j)^2/16^j/(2*j-1),j=1..L);
```

and then we again have a closed-form solution. The probability that the second player wins the game within k moves is

$$1 - \frac{2L+1}{16^L}\binom{2L}{L}^2, \quad \text{where } L = \left\lfloor \frac{k+1}{2} \right\rfloor.$$

The case $\{a, b\} = \{-1, 2\}$

For this case, each player is allowed to add two chips to his pile or remove one chip from his pile; each with probability $1/2$. The pile is allowed to have a negative number of chips. The first player who collects n chips wins the game.

Let $D(n, k)$ be the number of ways for a player to have $n-1$ or $n-2$ chips on his k^{th} move without ever collecting n chips (so the game still goes on).

The recurrence relation arises from whether the first move is $+2$ or -1,

$$D(n, k) = D(n-2, k-1) + D(n+1, k-1).$$

The case $\{a, b\} = \{-1, 2\}$

Lemma

The numbers $D(n, k)$ satisfy the following recurrence relation,

$$D(n, k) = D(n-1, k+1) - D(n-3, k)$$

with base cases $D(-1, k) = D(0, k) = 0$ and

$$D(1, 3m) = \frac{\binom{3m}{m}}{2m+1}, \quad D(1, 3m+1) = \frac{\binom{3m+1}{m+1}}{2m+1}, \quad D(1, 3m+2) = 0.$$

The case $\{a, b\} = \{-1, 2\}$

We have the following relation

$$r(n, k) = \frac{D(n, k-1)}{2^k}.$$

For each $n \geq 1$, we evaluate $\sum_{k=1}^{\infty} r^2(n, k)$ and apply Theorem 1 to find the winning probability of the second player.

However, this sum is not Gosper-summable, i.e., there is no nice closed-form formula for the partial sum.

The case $\{a, b\} = \{-1, 2\}$

We list their numerical values below.

n	$\sum_{k=1}^{\infty} r^2(n, k)$	p_n
1	0.3221721826105...	0.33891390869471156...
2	0.2886887304423...	0.35565563477884626...
3	0.1547549217692...	0.42262253911538507...
4	0.1241072133089...	0.43794639334553199...
5	0.0941564190484...	0.45292179047578731...
10	0.047917368748...	0.47604131562562199...
20	0.028469734522...	0.48576513273891113...
100	0.010952807500...	0.49452359624969611...

Open Problem

Find a recurrence relation for the terms $\sum_{k=1}^{\infty} r^2(n, k)$ (where $n \geq 0$).

A remark on Theorem 1

We would like to find an analog of Theorem 1 when players have the same set of moves (add a or b chips); but now the first player wins the game if he collects n_1 chips first and the second player wins the game if he collects n_2 chips first.

Let p_{n_1,n_2} be the probability that the second player collects n_2 chips before the first player collects n_1 chips. Then

$$p_{n_1,n_2} = \sum_{k=1}^{\infty} q(n_1,k) \cdot r(n_2,k). \tag{9}$$

p_{n_1,n_2} for the case $\{a, b\} = \{-1, 1\}$

For any fixed n, the probability $q(n, k)$ has a nice closed-form expression in k.

Previously, we obtained

$$q(1, 2m) = q(1, 2m-1) = \frac{\binom{2m}{m}}{4^m}.$$

Based on the closed-form expression of $q(n, k)$ and

$$r(n, k) = q(n, k-1) - q(n, k), \qquad k = 1, 2, \ldots,$$

We calculate values of p_{n_1,n_2}.

p_{n_1,n_2} for the case $\{a, b\} = \{-1, 1\}$

$n_1 \setminus n_2$	1	2	3	4
1	$\frac{\pi - 2}{\pi}$	$\frac{4 - \pi}{\pi}$	$\frac{10 - 3\pi}{\pi}$	$\frac{3\pi - 8}{3\pi}$
2	$\frac{2(\pi - 2)}{\pi}$	$\frac{3\pi - 8}{\pi}$	$\frac{2(10 - 3\pi)}{\pi}$	$\frac{3(16 - 5\pi)}{\pi}$
3	$\frac{3\pi - 2}{3\pi}$	$\frac{7\pi - 20}{\pi}$	$\frac{39\pi - 118}{3\pi}$	$\frac{296 - 93\pi}{3\pi}$
4	$\frac{8}{3\pi}$	$\frac{16 - 3\pi}{3\pi}$	$\frac{8(12\pi - 37)}{3\pi}$	$\frac{195\pi - 608}{3\pi}$

The winning probability of the second player, p_{n_1,n_2}

References

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