Integral linearization of shifted binomial powers

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Source and contribution. Conjecture 4.1 in T. A. Thanatipanonda, *Beyond Zudilin's Conjectured q-analog of Schmidt's problem*, arXiv:1403.4962v1, p. 6, asks for integer power coefficients and a single integer recursion valid for every positive power. Theorem 2.1 of that paper already explains how a product expansion implies the power recursion. We construct the required product coefficients for every nonnegative shift parameter d.

1. Exact statement and integral inverse

Here $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\binom{a}{b} = 0$ if $b > a$ for $a, b \in \mathbb{N}$. All binomial values below are viewed as integers. Fix $d \in \mathbb{N}$ and put $B_j(n) = \binom{n+dj}{j}$ for $j, n \in \mathbb{N}$.

Theorem 1. *For every $d, k \in \mathbb{N}$ there are integers $S_d(k, j, i)$, for all $i, j \in \mathbb{N}$, independent of n and of the power index r, such that $S_d(k, j, i) = 0$ whenever $j > i + k$, and*

$$B_k(n)B_i(n) = \sum_{j=0}^{i+k} S_d(k, j, i)B_j(n) \qquad (i, n \in \mathbb{N}). \tag{1}$$

Define $a^{(1)}_{d,k,j} = \delta_{j,k}$ and, for $r \geq 1$, define

$$a^{(r+1)}_{d,k,j} = \sum_{i=0}^{rk} S_d(k, j, i)a^{(r)}_{d,k,i}. \tag{2}$$

Then, for every $r \geq 1$ and $n \in \mathbb{N}$,

$$a^{(r)}_{d,k,j} = 0 \ (j > rk), \qquad \binom{n + dk}{k}^r = \sum_{j=0}^{rk} a^{(r)}_{d,k,j}\binom{n + dj}{j}. \tag{3}$$

Write $C_t(n) = \binom{n}{t}$ and $U_{t,v} = \binom{dt}{t-v}$ for $0 \leq v \leq t$. Vandermonde's identity gives the finite, unitriangular expansion

$$B_t(n) = \sum_{v=0}^{t} U_{t,v}C_v(n), \qquad U_{t,t} = 1. \tag{4}$$

Define integers $W_{t,j}$ recursively in t by

$$W_{t,j} = \begin{cases} 1, & j = t, \\ -\displaystyle\sum_{v=0}^{t-1} U_{t,v}W_{v,j}, & j \neq t. \end{cases} \tag{5}$$

An empty sum is zero. Strong induction gives $W_{t,j} = 0$ for $j > t$: each $W_{v,j}$ in the second line then vanishes because $v < t < j$. The same induction, rearranging (4), gives

$$\begin{aligned} C_t(n) &= B_t(n) - \sum_{v=0}^{t-1} U_{t,v}C_v(n) \\ &= B_t(n) - \sum_{j=0}^{t-1} \left(\sum_{v=0}^{t-1} U_{t,v}W_{v,j} \right) B_j(n) = \sum_{j=0}^{t} W_{t,j}B_j(n). \end{aligned}$$

Thus the inverse is integral; no division is needed.

2. Product coefficients and proof of the power identity

For all $n, p, q \in \mathbb{N}$,

$$\binom{n}{p}\binom{n}{q} = \sum_{u=0}^{p} \binom{q+u}{p}\binom{p}{u}\binom{n}{q+u}. \tag{6}$$

Indeed, the left side counts ordered pairs (P, Q) of subsets of an n-element set with $|P| = p$ and $|Q| = q$. Classify them by $u = |P \setminus Q|$, so $|P \cup Q| = q + u$. Choose the union, then choose P inside it, then choose the u elements of P absent from Q. These choices determine Q uniquely and give the three factors on the right. This also covers vanishing binomial coefficients.

Expanding B_k and B_i by (4), using (6), and then substituting the integral inverse gives the explicit construction

$$S_d(k, j, i) = \sum_{p=0}^{k} \sum_{q=0}^{i} \sum_{u=0}^{p} \binom{dk}{k-p}\binom{di}{i-q}\binom{q+u}{p}\binom{p}{u} W_{q+u,j}. \tag{7}$$

Every summand is an integer. Since $q + u \le i + k$ and $W_{q+u,j} = 0$ for $j > q + u$, this expression vanishes for $j > i + k$. Collecting the coefficient of $B_j(n)$ proves (1). The construction contains neither n nor r.

The initial row gives (3) at $r = 1$. Suppose the identity and support bound hold at $r \ge 1$. Multiplication by $B_k(n)$ gives

$$\begin{aligned} B_k(n)^{r+1} &= \sum_{i=0}^{rk} a^{(r)}_{d,k,i} B_k(n) B_i(n) \\ &= \sum_{i=0}^{rk} \sum_{j=0}^{(r+1)k} a^{(r)}_{d,k,i} S_d(k, j, i) B_j(n) \\ &= \sum_{j=0}^{(r+1)k} a^{(r+1)}_{d,k,j} B_j(n). \end{aligned}$$

The extension of the inner sum is valid by the product support bound. If $j > (r + 1)k$, then $j > i + k$ for every $i \le rk$, so every summand in (2) vanishes. This proves the next support bound and completes the induction, including $k = 0$ and $d = 0$.

A small check. At $d = 2$ and $k = i = 1$, the construction yields

$$(n+2)^2 = 2\binom{n+4}{2} - 3(n+2) - 2, \qquad (S_2(1,0,1), S_2(1,1,1), S_2(1,2,1)) = (-2, -3, 2).$$

The right side expands to $n^2 + 4n + 4$. Integer coefficients need not be nonnegative; the conjecture requires integrality.

3. What the Lean development checks

The linked Lean 4 module defines B, the coefficient recursion `a`, and the quantified proposition `claim`. The private theorems `integral_inverse` and `basis_product` establish the inverse and product identities above. The public theorem `result : claim` establishes the positive-power identity and both finite-support bounds for all natural parameters. The supplied verification record reports the standard axiom closure `propext`, `Classical.choice`, and `Quot.sound`; this note does not report a new Lean execution.

This proves Conjecture 4.1 as stated in arXiv v1. It does not prove Conjecture 4.2 (holonomicity and the claimed absence of a first-order closed form outside $d = 0, 1$), or formalize arbitrary shift functions. The abstract polynomial-basis uniqueness statement is also not part of the linked Lean theorem.

Source: `https://arxiv.org/abs/1403.4962v1`.

Lean formalization (for verification): `ShiftedBinomialPowerLinearization.lean`.

Fixed revision: `59c09d64d1050c396df54b58b4f410f09f0c25d0`.