

# Determinants Containing Powers of Generalized Fibonacci Numbers

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## Abstract

We study the determinants of matrices whose entries are powers of the Fibonacci numbers. We then extend the results to include entries that are powers of a second-order linear recurrence relation. These results motivate a fundamental identity of determinants whose entries are powers of linear polynomials. Finally, we discuss the determinants of matrices whose entries are products of the general second-order linear recurrence relations.

## 1 Introduction

In the first issue of the *Fibonacci Quarterly*, Alfred posed the following problem [1]:

Prove

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1},$$

where  $F_n$  is the  $n$ th Fibonacci number.

In the second volume of the *Fibonacci Quarterly*, Parker [7] posed a similar problem with the exponent of each entry changed to 4 and the dimension of the matrix changed to  $5 \times 5$ . These two results naturally suggest the following question: what would be the determinant of an analogous form where the dimension of the matrix and the exponent of each entry are arbitrary? Carlitz [4] answered this question by showing that the determinant of the form  $D(r, n) = \left| F_{n+i+j}^r \right|$ , where  $i, j = 0, 1, \dots, r$ , is given by

$$D(r, n) = (-1)^{(n+1)\binom{r+1}{2}} (F_1^r F_2^{r-1} \dots F_r)^2 \cdot \prod_{i=0}^r \binom{r}{i}. \quad (1)$$

In this paper we generalize the entries even further by considering the determinant of the form  $D(r, s, k, n) = \left| F_{s+k(n+i+j)}^r \right|$ , where  $i, j = 0, 1, \dots, r$ . We show that

$$D(r, s, k, n) = (-1)^{(s+kn+1)\binom{r+1}{2}} (F_k^r F_{2k}^{r-1} \dots F_{rk})^2 \cdot \prod_{i=0}^r \binom{r}{i}. \quad (2)$$

Carlitz based his proof of formula (1) on the Binet form of the Fibonacci numbers; whereas we employ the matrix methods, such as the factorization method of Krattenthaler, to prove formula (2). In addition, we require a generalized form of the Catalan identity, proved by Melham and Shannon [6]. In Section 2, we present an alternative proof of the generalized Catalan identity using a matrix representation of the sequence and the properties of the matrix multiplication. In Section 3, we then present the proof of formula (2) as a special case of the determinant with entries involving the powers of the numbers in a second-order linear recurrence with constant coefficients. In the last section, we present the determinant whose entries are the products of the numbers defined as a second-order linear recurrence with constant coefficients using the Desnanot-Jacobi identity. The methodology used for this work relies on a computer programming developed by the second author [8].

## 2 Generalized Catalan identity

The well-known Catalan identity states that for all nonnegative integers  $s$  and  $i$ ,

$$F_{s+i}^2 - F_s F_{s+2i} = (-1)^s F_i^2.$$

A generalization of this identity useful for this work is given by Melham and Shannon [6]. We shall, however, present an alternative proof of this generalization. For integers  $a, b, c_1$ , and  $c_2$  with  $c_2 \neq 0$ , let  $W_n = W_n(a, b; c_1, c_2)$  denote the second-order linear recurrence with constant coefficients, defined by

$$W_0 = a, W_1 = b \quad \text{and} \quad W_n = c_1 W_{n-1} + c_2 W_{n-2} \quad \text{for } n \geq 2.$$

With this notation, the Fibonacci sequence  $(F_n)$  and the Lucas sequence  $(U_n)$  correspond to  $F_n = W_n(0, 1; 1, 1)$  and  $U_n = W_n(0, 1; c_1, c_2)$ , respectively. Moreover, we can use this recurrence to extend the definition of a sequence to the terms with negative indices. Usually, we can explicitly find the relationship between the negative-indexed terms and the positive-indexed terms. For example, for the Fibonacci sequence and the Lucas sequence, we have

$$F_{-n} = (-1)^{n+1} F_n \quad \text{and} \quad U_{-n} = (-1)^{n+1} c_2^{-n} U_n \quad \text{for } n \geq 1.$$

**Proposition 1** (Generalized Catalan Identity). *Let  $W_n = W_n(a_0, a_1; c_1, c_2)$  and  $Y_n = W_n(b_0, b_1; c_1, c_2)$  be second-order linear recurrences. Then*

$$W_{s+i}Y_{s+j} - W_sY_{s+i+j} = (-c_2)^s(W_1Y_j - W_0Y_{j+1}) \cdot U_i, \quad (3)$$

for all integers  $s, j$ , and  $i$ .

*Proof.* The proof is by induction on  $i$ . For the case when  $i = 0$ , the identity is trivial. For the case when  $i = 1$ , we have

$$\begin{pmatrix} W_{s+1} & Y_{s+j+1} \\ W_s & Y_{s+j} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_s & Y_{s+j} \\ W_{s-1} & Y_{s+j-1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^s \begin{pmatrix} W_1 & Y_{j+1} \\ W_0 & Y_j \end{pmatrix} \quad \text{for } s \geq 0,$$

and

$$\begin{pmatrix} W_{s+1} & Y_{s+j+1} \\ W_s & Y_{s+j} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{s+2} & Y_{s+j+2} \\ W_{s+1} & Y_{s+j+1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^s \begin{pmatrix} W_1 & Y_{j+1} \\ W_0 & Y_j \end{pmatrix} \quad \text{for } s < 0,$$

where the second equality in both equations follows from repeated application of the matrix representation in the first equality. Taking the determinant of both sides of any one equation yields

$$W_{s+1}Y_{s+j} - W_sY_{s+j+1} = (-c_2)^s(W_1Y_j - W_0Y_{j+1}). \quad (4)$$

Now, consider two cases.

**Case  $i > 1$ :** Assume that the identity holds for some integers  $i - 1$  and  $i - 2$ . Then

$$\begin{aligned} W_{s+i}Y_{s+j} - W_sY_{s+i+j} &= \begin{vmatrix} W_{s+i} & Y_{s+i+j} \\ W_s & Y_{s+j} \end{vmatrix} \\ &= \begin{vmatrix} c_1W_{s+(i-1)} + c_2W_{s+(i-2)} & c_1Y_{s+(i-1)+j} + c_2Y_{s+(i-2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\ &= c_1 \begin{vmatrix} W_{s+(i-1)} & Y_{s+(i-1)+j} \\ W_s & Y_{s+j} \end{vmatrix} + c_2 \begin{vmatrix} W_{s+(i-2)} & Y_{s+(i-2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\ &= (-c_2)^s(W_1Y_j - W_0Y_{j+1})(c_1U_{i-1} + c_2U_{i-2}) \\ &= (-c_2)^s(W_1Y_j - W_0Y_{j+1})U_i, \end{aligned}$$

**Case  $i < 0$ :** Assume that the identity holds for some integers  $i + 1$  and  $i + 2$ . Then

$$\begin{aligned}
W_{s+i}Y_{s+j} - W_sY_{s+i+j} &= \begin{vmatrix} W_{s+i} & Y_{s+i+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= \begin{vmatrix} \frac{-c_1}{c_2}W_{s+(i+1)} + \frac{1}{c_2}W_{s+(i+2)} & \frac{-c_1}{c_2}Y_{s+(i+1)+j} + \frac{1}{c_2}Y_{s+(i+2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= \frac{-c_1}{c_2} \begin{vmatrix} W_{s+(i+1)} & Y_{s+(i+1)+j} \\ W_s & Y_{s+j} \end{vmatrix} + \frac{1}{c_2} \begin{vmatrix} W_{s+(i+2)} & Y_{s+(i+2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= (-c_2)^s (W_1Y_j - W_0Y_{j+1}) \left( \frac{-c_1}{c_2}U_{i+1} + \frac{1}{c_2}U_{i+2} \right) \\
&= (-c_2)^s (W_1Y_j - W_0Y_{j+1})U_i,
\end{aligned}$$

where we apply the induction hypothesis in the penultimate equality in both cases. Hence, the proof is complete.  $\square$

We note some special cases of Proposition 1 useful in later sections:

$$U_{s+i}U_{s+j} - U_sU_{s+i+j} = (-c_2)^s \cdot U_iU_j, \quad (5)$$

$$U_{s+i}W_{s+j} - U_sW_{s+i+j} = (-c_2)^s \cdot U_iW_j, \quad (6)$$

and

$$W_{s+i}W_{s+j} - W_sW_{s+i+j} = (-c_2)^s \cdot (W_1W_j - W_0W_{j+1})U_i = (-c_2)^s \cdot \Delta \cdot U_iU_j, \quad (7)$$

where  $\Delta = \begin{vmatrix} W_1 & W_2 \\ W_0 & W_1 \end{vmatrix} = a_1^2 - c_1a_0a_1 - c_2a_0^2$ .

We justify the second equality of (7) by applying Proposition 1 as follows: In (3), let  $Y_n = W_n$ , substitute  $j = 1$  and  $s = 0$ , respectively, and rename the index  $i$  by  $j$ .

The identity (7) can be restated as follows:

**Corollary 2.** *Let  $k, n, r, s$ , and  $t$  be integers. Then*

$$W_{s+k(n+t)} = A(t)W_{s+kn} + B(t)U_{s+k(n+r)}, \quad (8)$$

where  $A(t) = \frac{W_{k(t-r)}}{W_{-kr}}$  and  $B(t) = \frac{-(-c_2)^{-kr} \cdot \Delta \cdot U_{kt}}{W_{-kr}}$ .

*Proof.* Let integers  $k', n', r', s'$ , and  $t'$  be given. Applying (7) with  $s = -k'r'$ ,  $i = k't'$ , and  $j = s' + k'(n' + r')$ , we have

$$W_{-k'r'}W_{s'+k'(n'+t')} = W_{k'(t'-r')}W_{s'+k'n'} - (-c_2)^{-k'r'} \cdot \Delta \cdot U_{k't'}U_{s'+k'(n'+r')}.$$

Dividing by  $W_{-k'r'}$  on both sides and renaming the variables yield the identity (8), as required.  $\square$

### 3 Determinants involving powers of terms of second-order recurrence

Our goal in this section is to give the closed form of the determinant of the  $(r+1) \times (r+1)$  matrix whose entries are  $W_{s+k(n+i+j)}^r$ , where  $i, j = 0, 1, \dots, r$ , and  $s$  and  $k$  are any integers. This matrix is

$$\mathbb{A}_n^{s,k}(r) = \begin{pmatrix} W_{s+kn}^r & W_{s+k(n+1)}^r & \cdots & W_{s+k(n+r)}^r \\ W_{s+k(n+1)}^r & W_{s+k(n+2)}^r & \cdots & W_{s+k(n+r+1)}^r \\ \vdots & \vdots & \ddots & \vdots \\ W_{s+k(n+r)}^r & W_{s+k(n+r+1)}^r & \cdots & W_{s+k(n+2r)}^r \end{pmatrix}. \quad (9)$$

We begin with the following proposition on the determinant whose entries are some power of linear polynomials.

**Lemma 3.** *Let  $c_0, \dots, c_r$  and  $x_0, \dots, x_r$  be real numbers. Then*

$$\det((c_j x_i + 1)^r)_{0 \leq i, j \leq r} = \prod_{0 \leq i < j \leq r} (x_i - x_j) \prod_{0 \leq i < j \leq r} (c_i - c_j) \prod_{i=0}^r \binom{r}{i}. \quad (10)$$

*Proof.* We prove Lemma 3 by using the factorization method [5]. The determinant will be 0 if  $x_0$  is replaced by any  $x_i$  for  $0 < i \leq r$ , since some two rows of the matrix would be equal. This implies that  $(x_0 - x_i)$  is a factor of the determinant for each  $i = 1, 2, \dots, r$ . Similarly, we have that  $(x_1 - x_i)$  is a factor of the determinant for each  $i = 2, \dots, r$ , and so on. In a similar manner, we see that if  $c_0$  is replaced by any  $c_j$  for  $0 < j \leq r$ , then two columns of the matrix will be the same yielding the zero determinant. This implies that  $(c_0 - c_j)$  is a factor of the determinant for each  $j = 1, 2, \dots, r$ . Similarly, we have that  $(c_1 - c_j)$  is a factor of the determinant for each  $j = 2, \dots, r$ , and so on. Therefore

$$\prod_{0 \leq i < j \leq r} (x_i - x_j) \prod_{0 \leq i < j \leq r} (c_i - c_j) \quad (11)$$

is a factor of this determinant. As a function of  $x_i$  for some fixed  $i$  or a function of  $c_j$  for some fixed  $j$  the determinant is a polynomial of degree  $r$ . This implies that the factor (11) and the required determinant have the same degree. Therefore, we can write the determinant as

$$\det((c_j x_i + 1)^r)_{0 \leq i, j \leq r} = C \prod_{0 \leq i < j \leq r} (x_i - x_j) \prod_{0 \leq i < j \leq r} (c_i - c_j),$$

for some constant  $C$ . To find  $C$ , we compare on both sides the coefficients of the monomial

$$(c_r x_r)^r (c_{r-1} x_{r-1})^{r-1} \cdots (c_0 x_0)^0. \quad (12)$$

On the right-hand side, the monomial (12) appears as

$$\prod_{0 < i < j \leq r} (-x_j)(-c_j) = \prod_{0 < i < j \leq r} x_j c_j.$$

Hence, the coefficient of the monomial (12) on the right-hand side is just equal to  $C$ . We see that for each  $0 \leq i \leq r$ , the term  $(c_i x_i)^i$  appears in  $(c_i x_i + 1)^r$ . Hence, on the left-hand side, the monomial (12) appears as

$$\pm \prod_{0 \leq i \leq r} (c_i x_i + 1)^r.$$

By the definition of the determinant, the sign in front of the expression is determined by the parity of the identity permutation  $(0)(1) \cdots (r)$ . Since  $(0)(1) \cdots (r) = (01)(01)$ , it follows that the identity permutation is even. Hence, the sign is determined to be  $+$ . Since, for each  $0 \leq i \leq r$ , the coefficient of  $(c_i x_i)^i$  in  $(c_i x_i + 1)^r$  is  $\binom{r}{i}$ , it follows that

$$C = \prod_{i=0}^r \binom{r}{i}.$$

This completes the proof of Lemma 3. □

**Corollary 4.** *Let  $A(j), B(j), X_i, Y_i$  be real numbers for  $0 \leq i, j \leq r$ . Then*

$$\begin{aligned} & \det((A(j)X_i + B(j)Y_i)_{0 \leq i, j \leq r}) \\ &= \prod_{0 \leq i < j \leq r} (X_i Y_j - X_j Y_i) \prod_{0 \leq i < j \leq r} (A(i)B(j) - A(j)B(i)) \prod_{i=0}^r \binom{r}{i}. \end{aligned} \quad (13)$$

*Proof.* We prove in the case of  $B(j) \neq 0$  and  $Y_i \neq 0$  for all  $0 \leq i, j \leq r$ . Applying Lemma 3 with  $c_j = A(j)/B(j)$  and  $x_i = X_i/Y_i$  for all  $0 \leq i, j \leq r$  and clearing the denominators, we obtain (13). The proof of the other case when some of  $B(j)$  or  $Y_i$  are 0 follows from the fact that the determinant with polynomial entries is a continuous function. □

Thus, this allows us to prove one of the main results of this paper.

**Theorem 5.** *The determinant of the matrix  $\mathbb{A}_n^{s,k}(r)$  is given by*

$$\det \mathbb{A}_n^{s,k}(r) = (-1)^{(s+kn+1)\binom{r+1}{2}} \cdot c_2^{(s+kn)\binom{r+1}{2} + 2k\binom{r+1}{3}} \cdot \Delta^{\binom{r+1}{2}} \cdot \prod_{i=0}^r \binom{r}{i} U_{(i+1)k}^{2(r-i)}.$$

*Proof.* By (9), (8), (13), and (6) respectively, we have

$$\begin{aligned}
\det \mathbb{A}_n^{s,k}(r) &= \det(W_{s+k(n+i+j)}^r)_{0 \leq i, j \leq r} = \det((A(j)W_{s+k(n+i)} + B(j)U_{s+k(n+r+i)})^r)_{0 \leq i, j \leq r} \\
&= \prod_{0 \leq i < j \leq r} (W_{s+k(n+i)}U_{s+k(n+r+j)} - W_{s+k(n+j)}U_{s+k(n+r+i)}) \prod_{0 \leq i < j \leq r} (A(i)B(j) - A(j)B(i)) \prod_{i=0}^r \binom{r}{i} \\
&= \prod_{0 \leq i < j \leq r} ((-c_2)^{s+k(n+r+i)}U_{k(j-i)}W_{-kr}) \prod_{0 \leq i < j \leq r} \left( \frac{-\Delta \cdot (-c_2)^{-kr}}{W_{-kr}^2} (-c_2)^{ki}U_{k(j-i)}W_{-kr} \right) \prod_{i=0}^r \binom{r}{i} \\
&= \prod_{0 \leq i < j \leq r} \left( -\Delta \cdot (-c_2)^{s+k(n+2i)}U_{k(j-i)}^2 \right) \prod_{i=0}^r \binom{r}{i},
\end{aligned}$$

Rearranging the last expression allows us to obtain the desired identity.  $\square$

*Remark 6.* By letting  $W_n = F_n$  in Theorem 5 and noting that for the Fibonacci sequence  $c_2 = 1$  and  $\Delta = 1$ , we then derive (2).

## 4 Determinants involving products of terms of second-order recurrence

The following lemma was mentioned by Krattenthaler [5] as part of the factorization method. We provide a different proof of this lemma using the Desnanot-Jacobi identity [3].

**Lemma 7.** *Let  $X_0, \dots, X_r, D_1, \dots, D_r$ , and  $E_1, \dots, E_r$  be indeterminates. Then*

$$\det \left( \prod_{\ell=j+1}^r (X_i + D_\ell) \cdot \prod_{m=1}^j (X_i + E_m) \right)_{0 \leq i, j \leq r} = \prod_{0 \leq i < j \leq r} (X_j - X_i) \cdot \prod_{1 \leq i \leq j \leq r} (D_j - E_i).$$

An alternative way of writing this identity would be

$$\begin{aligned}
\det \left( \prod_{\ell=j+1}^r (A(d_\ell)X_i + B(e_\ell)Y_i) \cdot \prod_{m=1}^j (A(e_m)X_i + B(e_m)Y_i) \right)_{0 \leq i, j \leq r} \\
= \prod_{0 \leq i < j \leq r} (X_i Y_j - X_j Y_i) \prod_{1 \leq i \leq j \leq r} (B(e_i)A(d_j) - A(e_i)B(d_j)). \quad (14)
\end{aligned}$$

The main result of this section is as follows:

**Theorem 8.** Let  $d_1, \dots, d_r$  and  $e_1, \dots, e_r$  be sequences of integers. Then

$$\det \left( \prod_{\ell=j+1}^r W_{s+k(n+i+d_\ell)} \cdot \prod_{m=1}^j W_{s+k(n+i+e_m)} \right)_{0 \leq i, j \leq r} \\ = (-\Delta)^{\binom{r+1}{2}} \cdot (-c_2)^{(s+kn)\binom{r+1}{2} + k\binom{r+1}{3}} \cdot \prod_{\ell=1}^r U_{k\ell}^{r+1-\ell} \prod_{1 \leq i \leq j \leq r} (-c_2)^{kd_j} U_{k(e_i-d_j)}.$$

*Proof.* We respectively apply the identities (8), (14), and (7), and the details of the proof are similar to those of Theorem 5.  $\square$

The following slight variation of the result by Alfred [2] called *basic power determinant* is a special case of this theorem.

**Corollary 9.** Let  $r, s$ , and  $k$  be integers with  $r \geq 0$ . Then

$$\begin{vmatrix} F_s^r & F_s^{r-1} F_{s+k} & \cdots & F_{s+k}^r \\ F_{s+k}^r & F_{s+k}^{r-1} F_{s+2k} & \cdots & F_{s+2k}^r \\ \vdots & \vdots & \ddots & \vdots \\ F_{s+rk}^r & F_{s+rk}^{r-1} F_{s+(r+1)k} & \cdots & F_{s+(r+1)k}^r \end{vmatrix} = (-1)^{(s+1)\binom{r+1}{2} + k\binom{r+1}{3}} F_k^{\binom{r+1}{2}} \prod_{\ell=1}^r F_{k\ell}^{r+1-\ell}.$$

*Proof.* This identity follows immediately from Theorem 8 by letting  $d_1 = d_2 = \dots = d_r = 0$ ,  $e_1 = e_2 = \dots = e_r = 1$ ,  $n = 0$  together with the recurrence and the initial values of the Fibonacci numbers.  $\square$

Another interesting case arises when we let  $(d_j)$  and  $(e_j)$  in Theorem 8 be in some specific forms.

**Corollary 10.** Let  $s, k, n$ , and  $p$  be integers. Let  $d_j = p-1+j$  and  $e_j = j-1$  for  $1 \leq j \leq r$ . Then

$$\det \left( \prod_{\ell=j+1}^r W_{s+k(n+i+d_\ell)} \cdot \prod_{m=1}^j W_{s+k(n+i+e_m)} \right)_{0 \leq i, j \leq r} \\ = \Delta^{\binom{r+1}{2}} (-c_2)^{(s+kn)\binom{r+1}{2} + 2k\binom{r+1}{3}} \cdot \prod_{\ell=1}^r U_{k\ell}^{r+1-\ell} \cdot \prod_{\ell=0}^{r-1} U_{k(p+\ell)}^{-\ell}.$$

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