

# LIBRARY VALUES OF TOADS AND FROGS

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ABSTRACT. We list the values of all positions of the classes  $\square F$ ,  $\square\square F$ ,  $\square\square\square F$ ,  $\square\square\square\square F$ ,  $\square\square\square\square\square F$ ,  $\square FF$ ,  $\square\square FF$ ,  $\square\square\square FF$ ,  $\square FFF$ ,  $TF$  of combinatorial game, Toads and Frogs. We hope this will make a good reference for the future works relate to this game.

## 1. INTRODUCTION.

We present the values of Toads and Frogs as a reference to the paper “A Symbolic Finite-state approach for Automated Proving of Theorems in Combinatorial Game Theory” (See [3]). In section 2, we show the values of all positions of the classes  $\square F$ ,  $\square\square F$ ,  $\square\square\square F$ ,  $\square\square\square\square F$ ,  $\square\square\square\square\square F$ . In section 3, we show the values of all positions of the classes  $\square FF$ ,  $\square\square FF$ ,  $\square\square\square FF$ . In section 4, we show the values of all positions of the class  $\square FFF$ . In section 5, we show the values of all positions of the class  $TF$ .

All the values in this paper has already been proved. For the classes with one frog:  $\square F$ ,  $\square\square F$ ,  $\square\square\square F$ ,  $\square\square\square\square F$ ,  $\square\square\square\square\square F$ , we have an automated program to conjecture and prove everything automatically (See [3]). For the classes with two frogs and three frogs,  $\square FF$ ,  $\square\square FF$ ,  $\square\square\square FF$ ,  $\square FFF$ , we use human ingenuity to do conjectures but use the same computer program that we wrote to prove the conjectures. For the class  $TF$ , all the values are already proved by hand (See [3]).

Note that for the class with exactly one blank and specific number of frogs, we already knew all the values in these classes (See [4]).

The readers only need the minimum knowledge of combinatorial game theory to understand the paper. Please refer to the first two chapters of [1] for the necessary background.

The only notations we use are  $\ast (= \{0 \mid 0\})$  and  $n \ast (= \{n \mid n\})$ . We will not use any shorthand notation like  $\uparrow$ ,  $\uparrow\uparrow$ , etc.

**Note:** We apply the Terminal Toads Theorem to simplify the lists of the values of the formulas.

**The Terminal Toads Theorem:** Let  $X$  be any position. Then  $X T \square^n = X \square^n + n$ .

## 2. RESULT OF CLASS WITH ONE FROGS.

ClassA11:  $\square F$ Let  $f(a, b)$  be the value of  $T^a \square T^b F$ Let  $g(a)$  be the value of  $T^a F \square$ 

Values:

$$\begin{aligned}
f(0, 0) &= -1. \\
f(a, 0) &= \{\{a - 2 \mid 1\} \mid 0\} \quad , \quad a \geq 1. \\
f(a, 1) &= \{a - 1 \mid 1\} \quad , \quad a \geq 0. \\
f(a, b) &= a \quad , \quad a \geq 0, b \geq 2. \\
g(a) &= 0 \quad , \quad a \geq 0.
\end{aligned}$$

ClassA21:  $\square \square F$ Let  $f(a, b, c)$  be the value of  $T^a \square T^b \square T^c F$ Let  $g(a, b)$  be the value of  $T^a \square T^b F \square$ Let  $h(a)$  be the value of  $T^a F \square \square$ 

Values:

$$\begin{aligned}
f(0, 0, 0) &= -2. \\
f(a, 0, 0) &= a - 1 \quad , \quad a \geq 1. \\
f(0, 1, 0) &= -\frac{1}{2}. \\
f(a, 1, 0) &= a^* \quad , \quad a \geq 1. \\
f(a, b, 0) &= \{\{2a + b - 2 \mid a + 1\} \mid a\} \quad , \quad a \geq 0, b \geq 2. \\
f(0, 0, 1) &= -1. \\
f(a, 0, 1) &= a \quad , \quad a \geq 1. \\
f(a, b, 1) &= \{2a + b - 1 \mid a + 1\} \quad , \quad a \geq 0, b \geq 1. \\
f(a, b, c) &= 2a + b \quad , \quad a \geq 0, b \geq 0, c \geq 2. \\
g(0, 0) &= -1. \\
g(a, 0) &= a - \frac{1}{2} \quad , \quad a \geq 1. \\
g(a, b) &= a \quad , \quad a \geq 0, b \geq 1. \\
h(a) &= a \quad , \quad a \geq 0.
\end{aligned}$$

ClassA31:  $\square \square \square F$ Let  $f(a, b, c, d)$  be the value of  $T^a \square T^b \square T^c \square T^d F$ .Let  $g(a, b, c)$  be the value of  $T^a \square T^b \square T^c F \square$ .Let  $h(a, b)$  be the value of  $T^a \square T^b F \square \square$ .Let  $i(a)$  be the value of  $T^a F \square \square \square$ .

Values:

$$\begin{aligned}
f(0, 0, 0, 0) &= -3. \\
f(a, 0, 0, 0) &= (2a - 2)* && , a \geq 1. \\
f(a, b, 0, 0) &= 2a + b - 1 && , a \geq 0, b \geq 1. \\
f(a, 0, 1, 0) &= 2a - 1 && , a \geq 0. \\
f(a, b, 1, 0) &= (2a + b)* && , a \geq 0, b \geq 1. \\
f(a, b, c, 0) &= \{\{3a + 2b + c - 2 \mid 2a + b + 1\} \mid 2a + b\} && , a \geq 0, b \geq 0, c \geq 2. \\
f(0, 0, 0, 1) &= -2. \\
f(a, 0, 0, 1) &= (2a - 1)* && , a \geq 1. \\
f(a, b, 0, 1) &= 2a + b && , a \geq 0, b \geq 1. \\
f(a, b, c, 1) &= \{3a + 2b + c - 1 \mid 2a + b + 1\} && , a \geq 0, b \geq 0, c \geq 1. \\
f(a, b, c, d) &= 3a + 2b + c && , a \geq 0, b \geq 0, c \geq 0, d \geq 2. \\
\\
g(a, 0, 0) &= 2a - 2 && , a \geq 0. \\
g(a, b, 0) &= (2a + b - 1)* && , a \geq 0, b \geq 1. \\
g(a, b, c) &= 2a + b && , a \geq 0, b \geq 0, c \geq 1. \\
\\
h(a, b) &= 2a + b - 1 && , a \geq 0, b \geq 0. \\
\\
i(a) &= 2a && , a \geq 0.
\end{aligned}$$

ClassA41:  $\square\square\square\square F$

Let  $f(a, b, c, d, e)$  be the value of  $T^a \square T^b \square T^c \square T^d \square T^e F$ .

Let  $g(a, b, c, d)$  be the value of  $T^a \square T^b \square T^c \square T^d F \square$ .

Let  $h(a, b, c)$  be the value of  $T^a \square T^b \square T^c F \square \square$ .

Let  $i(a, b)$  be the value of  $T^a \square T^b F \square \square \square$ .

Let  $j(a)$  be the value of  $T^a F \square \square \square \square$ .

Values:

$$\begin{aligned}
f(0, 0, 0, 0, 0) &= -4. \\
f(a, 0, 0, 0, 0) &= 3a - 3 && , a \geq 1. \\
f(0, 1, 0, 0, 0) &= -\frac{1}{2}. \\
f(a, 1, 0, 0, 0) &= 3a - \frac{1}{4} && , a \geq 1. \\
f(a, b, 0, 0, 0) &= (3a + 2b - 2)* && , a \geq 0, b \geq 2. \\
f(0, 0, 1, 0, 0) &= -1. \\
f(a, 0, 1, 0, 0) &= 3a - \frac{1}{2} && , a \geq 1. \\
f(a, b, c, 0, 0) &= 3a + 2b + c - 1 && , a \geq 0, b \geq 1, c = 1 \text{ or } a \geq 0, b \geq 0, c \geq 2. \\
f(0, 0, 0, 1, 0) &= -2. \\
f(a, b, 0, 1, 0) &= 3a + 2b - 1 && , a \geq 1, b = 0 \text{ or } a \geq 0, b \geq 1. \\
f(0, 0, 1, 1, 0) &= \frac{1}{2}. \\
f(a, 0, 1, 1, 0) &= 3a + \frac{3}{4} && , a \geq 1. \\
f(a, b, c, 1, 0) &= (3a + 2b + c)* && , a \geq 0, b \geq 1, c = 1 \text{ or } a \geq 0, b \geq 0, c \geq 2. \\
f(0, 0, 0, 2, 0) &= \{ * \mid 0 \}. \\
f(a, 0, 0, 2, 0) &= \{\{4a \mid 3a + \frac{1}{2}\} \mid 3a\} && , a \geq 1.
\end{aligned}$$

$$f(a, b, c, d, 0) = \{\{4a + 3b + 2c + d - 2 \mid 3a + 2b + c + 1\} \mid 3a + 2b + c\}$$

$$\begin{aligned} & , a \geq 0, b \geq 1, c = 0, d = 2 \\ & \text{or } a \geq 0, b \geq 0, c \geq 1, d = 2 \\ & \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 3. \end{aligned}$$

$$\begin{aligned} f(0, 0, 0, 0, 1) &= -3. \\ f(a, 0, 0, 0, 1) &= 3a - 2, \quad a \geq 1. \\ f(0, 1, 0, 0, 1) &= \frac{1}{2}. \\ f(a, 1, 0, 0, 1) &= 3a + \frac{3}{4}, \quad a \geq 1. \\ f(a, b, 0, 0, 1) &= (3a + 2b - 1)*, \quad a \geq 0, b \geq 2. \\ f(0, 0, 1, 0, 1) &= 0. \\ f(a, 0, 1, 0, 1) &= 3a + \frac{1}{2}, \quad a \geq 1. \\ f(a, b, c, 0, 1) &= 3a + 2b + c, \quad a \geq 0, b \geq 1, c = 1 \text{ or } a \geq 0, b \geq 0, c \geq 2. \\ f(0, 0, 0, 1, 1) &= *. \\ f(a, 0, 0, 1, 1) &= \{4a \mid 3a + \frac{1}{2}\}, \quad a \geq 1. \end{aligned}$$

$$f(a, b, c, d, 1) = \{4a + 3b + 2c + d - 1 \mid 3a + 2b + c + 1\}$$

$$\begin{aligned} & , a \geq 0, b \geq 1, c = 0, d = 1 \text{ or} \\ & a \geq 0, b \geq 0, c \geq 1, d = 1 \text{ or} \\ & a \geq 0, b \geq 0, c \geq 0, d \geq 2. \end{aligned}$$

$$f(a, b, c, d, e) = 4a + 3b + 2c + d, \quad a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 2.$$

$$\begin{aligned} g(0, 0, 0, 0) &= -3. \\ g(a, 0, 0, 0) &= 3a - \frac{5}{2}, \quad a \geq 1. \\ g(a, b, 0, 0) &= 3a + 2b - 2, \quad a \geq 0, b \geq 1. \\ g(0, 0, 1, 0) &= -\frac{1}{2}. \\ g(a, 0, 1, 0) &= 3a - \frac{1}{4}, \quad a \geq 1. \\ g(a, b, c, 0) &= (3a + 2b + c - 1)*, \quad a \geq 0, b \geq 1, c = 1 \\ & \text{or } a \geq 0, b \geq 0, c \geq 2. \end{aligned}$$

$$\begin{aligned} g(0, 0, 0, 1) &= -1. \\ g(a, 0, 0, 1) &= 3a - \frac{1}{2}, \quad a \geq 1. \\ g(a, b, c, d) &= 3a + 2b + c, \quad a \geq 0, b \geq 1, c = 0, d = 1 \\ & \text{or } a \geq 0, b \geq 0, c \geq 1, d = 1 \\ & \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 2. \end{aligned}$$

$$\begin{aligned} h(a, 0, 0) &= 3a - 2, \quad a \geq 0. \\ h(a, b, 0) &= 3a + 2b - \frac{3}{2}, \quad a \geq 0, b \geq 1. \\ h(a, b, c) &= 3a + 2b + c - 1, \quad a \geq 0, b \geq 0, c \geq 1. \end{aligned}$$

$$i(a, b) = 3a + 2b - 1, \quad a \geq 0, b \geq 0.$$

$$j(a) = 3a, \quad a \geq 0.$$

ClassA51:  $\square\square\square\square\square F$

Let  $f(a, b, c, d, e, l)$  be the value of  $T^a \square T^b \square T^c \square T^d \square T^e \square T^l F$ .

Let  $g(a, b, c, d, e)$  be the value of  $T^a \square T^b \square T^c \square T^d \square T^e F \square$ .

Let  $h(a, b, c, d)$  be the value of  $T^a \square T^b \square T^c \square T^d F \square \square$ .

Let  $i(a, b, c)$  be the value of  $T^a \square T^b \square T^c F \square \square \square$ .

Let  $j(a, b)$  be the value of  $T^a \square T^b F \square \square \square \square$ .

Let  $k(a)$  be the value of  $T^a F \square \square \square \square \square$ .

Values:

$$\begin{aligned}
 f(0, 0, 0, 0, 0, 0) &= -5. \\
 f(a, 0, 0, 0, 0, 0) &= (4a - 4)* && , a \geq 1. \\
 f(a, b, 0, 0, 0, 0) &= 4a + 3b - 3 && , a \geq 0, b \geq 1. \\
 f(a, b, 1, 0, 0, 0) &= 4a + 3b - 1 && , a \geq 0, b \geq 0. \\
 f(a, b, c, 0, 0, 0) &= (4a + 3b + 2c - 2)* && , a \geq 0, b \geq 0, c \geq 2. \\
 f(a, 0, 0, 1, 0, 0) &= 4a - 2 && , a \geq 0. \\
 f(a, b, 0, 1, 0, 0) &= (4a + 3b - 1)* && , a \geq 0, b \geq 1. \\
 f(a, b, c, d, 0, 0) &= 4a + 3b + 2c + d - 1 && , a \geq 0, b \geq 0, c \geq 1, d = 1 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 2. \\
 f(0, 0, 0, 0, 1, 0) &= -3. \\
 f(a, 0, 0, 0, 1, 0) &= (4a - 2)* && , a \geq 1. \\
 f(a, b, c, 0, 1, 0) &= 4a + 3b + 2c - 1 && , a \geq 0, b \geq 1, c = 0 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 1, . \\
 f(a, b, 0, 1, 1, 0) &= 4a + 3b && , a \geq 0, b \geq 0. \\
 f(a, b, c, d, 1, 0) &= (4a + 3b + 2c + d)* && , a \geq 0, b \geq 0, c \geq 1, d = 1 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 2. \\
 f(a, 0, 0, 0, 2, 0) &= 4a - 1 && , a \geq 0. \\
 f(a, b, 0, 0, 2, 0) &= (4a + 3b)* && , a \geq 0, b \geq 1.
 \end{aligned}$$

$$\begin{aligned}
 f(a, b, c, d, e, 0) &= \{ \{ 5a + 4b + 3c + 2d + e - 2 \mid 4a + 3b + 2c + d + 1 \} \mid 4a + 3b + 2c + d \} \\
 &&& , a \geq 0, b \geq 0, c \geq 1, d = 0, e = 2 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 1, e = 2 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 3.
 \end{aligned}$$

$$f(a, b, c, d, 0, 1) = f(a, b, c, d, 0, 0) + 1 \quad , a \geq 0, b \geq 0, c \geq 0, d \geq 0.$$

$$\begin{aligned}
 f(a, 0, 0, 0, 1, 1) &= \{ 5a \mid 4a - 1 \} && , a \geq 0. \\
 f(a, b, 0, 0, 1, 1) &= \{ 5a + 4b \mid (4a + 3b)* \} && , a \geq 0, b \geq 1.
 \end{aligned}$$

$$\begin{aligned}
 f(a, b, c, d, e, 1) &= \{ 5a + 4b + 3c + 2d + e - 1 \mid 4a + 3b + 2c + d + 1 \} , \\
 &&& a \geq 0, b \geq 0, c \geq 1, d = 0, e = 1 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 1, e = 1 \\
 &&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 2 .
 \end{aligned}$$

$$f(a, b, c, d, e, l) = 5a + 4b + 3c + 2d + e, \quad a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 0, l \geq 2.$$

$$\begin{aligned}
g(a, 0, 0, 0, 0) &= 4a - 4 && , a \geq 0. \\
g(a, b, 0, 0, 0) &= (4a + 3b - 3)* && , a \geq 0, b \geq 1. \\
g(a, b, c, 0, 0) &= 4a + 3b + 2c - 2 && , a \geq 0, b \geq 0, c \geq 1. \\
g(a, b, 0, 1, 0) &= 4a + 3b - 1 && , a \geq 0, b \geq 0. \\
g(a, b, c, d, 0) &= (4a + 3b + 2c + d - 1)* && , a \geq 0, b \geq 0, c \geq 1, d = 1 \\
&&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 2. \\
\\
g(a, 0, 0, 0, 1) &= 4a - 2 && , a \geq 0. \\
g(a, b, 0, 0, 1) &= (4a + 3b - 1)* && , a \geq 0, b \geq 1. \\
g(a, b, c, d, e) &= 4a + 3b + 2c + d && , a \geq 0, b \geq 0, c \geq 1, d = 0, e = 1 \\
&&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 1, e = 1 \\
&&& \text{or } a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 2. \\
\\
h(a, b, 0, 0) &= 4a + 3b - 3 && , a \geq 0, b \geq 0. \\
h(a, b, c, 0) &= (4a + 3b + 2c - 2)* && , a \geq 0, b \geq 0, c \geq 1. \\
h(a, b, c, d) &= 4a + 3b + 2c + d - 1 && , a \geq 0, b \geq 0, c \geq 0, d \geq 1. \\
\\
i(a, b, c) &= 4a + 3b + 2c - 2 && , a \geq 0, b \geq 0, c \geq 0. \\
\\
j(a, b) &= 4a + 3b - 1 && , a \geq 0, b \geq 0. \\
\\
k(a) &= 4a && , a \geq 0.
\end{aligned}$$

### 3. RESULT OF CLASS WITH TWO FROGS.

ClassA12:  $\square FF$

Let  $f(a, b, c)$  be the value of  $T^a \square T^b FT^c F$

Let  $g(a, b, c)$  be the value of  $T^a FT^b \square T^c F$

Let  $h(a, b)$  be the value of  $T^a FT^b F \square$

Values:

$$\begin{aligned}
f(0, 0, 0) &= -2. \\
f(0, 0, c) &= -1 && , c \geq 1. \\
f(1, 0, 0) &= \{0 \mid -\frac{1}{2}\}. \\
f(a, 0, c) &= \{\{a - 2 \mid \{0 \mid c\}\} \mid 0\} && , a = 1, c \geq 1 \text{ or } a \geq 2, c \geq 0. \\
f(a, 1, c) &= \{a - 1 \mid \{0 \mid c\}\} && , a \geq 0, c \geq 0. \\
f(a, b, c) &= a && , a \geq 0, b \geq 2, c \geq 0.
\end{aligned}$$

$$\begin{aligned}
g(0,0,0) &= -1. \\
g(1,0,0) &= -\frac{1}{2}. \\
g(a,0,0) &= \{\{\{a-3 \mid \frac{1}{2}\} \mid 0\} \mid 0\} \quad , \quad a \geq 2. \\
g(a,b,0) &= \{\{b-2 \mid 1\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 1. \\
g(a,b,1) &= \{b-1 \mid 1\} \quad , \quad a \geq 0, \quad b \geq 0. \\
g(a,b,c) &= b \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \\
\\
h(a,b) &= 0 \quad , \quad a \geq 0, \quad b \geq 0.
\end{aligned}$$

ClassA22:  $\square\square\text{FF}$

Let  $f(a,b,c,d)$  be the value of  $\text{T}^a\square\text{T}^b\square\text{T}^c\text{FT}^d\text{F}$ .

Let  $g(a,b,c,d)$  be the value of  $\text{T}^a\square\text{T}^b\text{FT}^c\square\text{T}^d\text{F}$ .

Let  $h(a,b,c,d)$  be the value of  $\text{T}^a\text{FT}^b\square\text{T}^c\square\text{T}^d\text{F}$ .

Let  $i(a,b,c)$  be the value of  $\text{T}^a\square\text{T}^b\text{FT}^c\text{F}\square$ .

Let  $j(a,b,c)$  be the value of  $\text{T}^a\text{FT}^b\square\text{T}^c\text{F}\square$ .

Let  $k(a,b)$  be the value of  $\text{T}^a\text{FT}^b\text{F}\square\square$ .

**Note:** In  $f, g, h$  we omit the case when  $d \geq 2$  since it will reduce to ClassA21.

Values:

**For  $d = 0$**

$$\begin{aligned}
f(0,0,0,0) &= -4. \\
f(1,0,0,0) &= -1. \\
f(2,0,0,0) &= *. \\
f(a,0,0,0) &= \{a - \frac{5}{2} \mid 0\} \quad , \quad a \geq 3. \\
f(0,1,0,0) &= \frac{-3}{2}. \\
f(1,1,0,0) &= 0. \\
f(2,1,0,0) &= \{1 \mid \{\frac{1}{2} \mid 0\}\}. \\
f(a,1,0,0) &= a - \frac{3}{2} \quad , \quad a \geq 3. \\
f(0,2,0,0) &= \{\{0 \mid *\} \mid \frac{-1}{4}\}. \\
f(1,2,0,0) &= \frac{1}{2}*. \\
f(a,2,0,0) &= a - 1 \quad , \quad a \geq 2. \\
f(a,b,0,0) &= a* \quad , \quad a \geq 0, \quad b \geq 3.
\end{aligned}$$

$$\begin{aligned}
f(0,0,1,0) &= -2. \\
f(1,0,1,0) &= *. \\
f(2,0,1,0) &= 1. \\
f(3,0,1,0) &= 1. \\
f(a,0,1,0) &= \{(a-2)* \mid 1*\} \quad , \quad a \geq 4. \\
f(0,1,1,0) &= \{0 \mid *\}. \\
f(1,1,1,0) &= \{2 \mid \frac{1}{2}*\}. \\
f(a,1,1,0) &= \{2a \mid \{(a-1)* \mid 1*\}\} \quad , \quad a \geq 2. \\
f(a,b,1,0) &= \{2a + b - 1 \mid a*\} \quad , \quad a \geq 0, \quad b \geq 2.
\end{aligned}$$

**For  $d = 1$**

$$\begin{aligned}
f(0, 0, 0, 1) &= -3. \\
f(a, 0, 0, 1) &= \{a - 2 \mid 1\} && , a \geq 1. \\
f(0, 1, 0, 1) &= -\frac{1}{2}. \\
f(a, 1, 0, 1) &= \{a - \frac{1}{2} \mid \{a - 1 \mid 1\}\} && , a \geq 1. \\
f(a, 2, 0, 1) &= \{\{2a \mid \{a \mid \{a \mid 2\}\}\} \mid a\} && , a \geq 0. \\
f(a, b, 0, 1) &= \{\{2a + b - 2 \mid a + \frac{1}{2}\} \mid a\} && , a \geq 0, b \geq 3. \\
f(0, 0, 1, 1) &= -1. \\
f(a, 0, 1, 1) &= a - \frac{1}{2} && , a \geq 1 \\
f(a, 1, 1, 1) &= \{2a \mid \{a \mid \{a \mid 2\}\}\} && , a \geq 0. \\
f(a, b, 1, 1) &= \{2a + b - 1 \mid a + \frac{1}{2}\} && , a \geq 0, b \geq 2. \\
f(a, b, c, d) &= 2a + b && , a \geq 0, b \geq 0, c \geq 2, d \geq 0.
\end{aligned}$$

**For  $d = 0$**

**First  $a = 0, b = 0.$**

$$\begin{aligned}
g(0, 0, 0, 0) &= -3. \\
g(0, 0, 1, 0) &= -\frac{3}{2}. \\
g(0, 0, c, 0) &= \{\{c - 3 \mid 0\} \mid -1\} && , c \geq 2 \\
g(0, 0, 0, 1) &= -2. \\
g(0, 0, c, 1) &= \{c - 2 \mid 0\} && , c \geq 1 \\
g(0, 0, c, d) &= c - 1 && , c \geq 0, d \geq 2.
\end{aligned}$$

**Second  $c = 0, d = 0.$**

$$\begin{aligned}
g(1, 0, 0, 0) &= (-1) * . \\
g(a, 0, 0, 0) &= 0 && , a \geq 2. \\
g(0, 1, 0, 0) &= -1. \\
g(1, 1, 0, 0) &= * . \\
g(a, 1, 0, 0) &= \{a - \frac{3}{2} \mid 0\} && , a \geq 2. \\
g(0, 2, 0, 0) &= -\frac{1}{4}. \\
g(a, 2, 0, 0) &= a - \frac{1}{2} && , a \geq 1. \\
g(a, 3, 0, 0) &= \{\{\{2a \mid \{a \mid \{a \mid 2\}\}\} \mid a\} \mid a\} && , a \geq 0. \\
g(a, b, 0, 0) &= \{\{\{2a + b - 3 \mid a + \frac{1}{2}\} \mid a\} \mid a\} && , a \geq 0, b \geq 4.
\end{aligned}$$

**Third  $b = 0, d = 0.$**

$$\begin{aligned}
g(1, 0, 1, 0) &= 0. \\
g(2, 0, 1, 0) &= \{1 \mid \{\frac{1}{2} \mid 0\}\}. \\
g(a, 0, 1, 0) &= 1* && , a \geq 3. \\
g(a, 0, c, 0) &= \{\{a + c - \frac{5}{2} \mid \{\{a - 1 \mid 2\} \mid 1\}\} \mid \{\{a - 2 \mid 1\} \mid 0\}\} && , a \geq 1, c \geq 2.
\end{aligned}$$



**Fourth**  $b = 1, d = 0$ .

$$\begin{aligned} g(0, 1, 1, 0) &= *. \\ g(1, 1, 1, 0) &= \frac{1}{2} *. \\ g(a, 1, 1, 0) &= \{(a-1)* | 1*\} \quad , \quad a \geq 2. \\ g(a, 1, c, 0) &= \{\{a+c-2 | \{a | 2\}\} | \{a-1 | 1\}, 2\} \quad , \quad a \geq 0, \quad c \geq 2. \end{aligned}$$

**Last**  $b \geq 2, d = 0$ .

$$g(a, b, c, 0) = \{\{a+c-2 | a+1\} | a\} \quad , \quad a \geq 0, \quad b \geq 2, \quad c \geq 1.$$

**Now for**  $d = 1$

**First**  $b = 0, d = 1$ .

$$\begin{aligned} g(1, 0, 0, 1) &= *. \\ g(a, 0, 0, 1) &= 1 \quad , \quad a \geq 2. \\ g(a, 0, c, 1) &= \{\{a+c-\frac{3}{2} | \{a-1 | 2\} | 1\}\} \quad , \quad a \geq 1, \quad c \geq 1. \end{aligned}$$

**Second**  $b = 1, d = 1$ .

$$\begin{aligned} g(a, 1, 0, 1) &= \{a-1 | 1\} \quad , \quad a \geq 0. \\ g(a, 1, c, 1) &= \{a+c-1 | \{a | 2\}\} \quad , \quad a \geq 0, \quad c \geq 1. \end{aligned}$$

**Last**  $b \geq 2, d = 1$ .

$$g(a, b, c, 1) = \{a+c-1 | a+1\} \quad , \quad a \geq 0, \quad b \geq 2, \quad c \geq 0.$$

$$\begin{aligned} h(0, 0, 0, 0) &= -2. \\ h(1, 0, 0, 0) &= -1. \\ h(2, 0, 0, 0) &= *. \\ h(3, 0, 0, 0) &= \{\frac{1}{2} | 0\}. \\ h(a, 0, 0, 0) &= \{1* | 0\} \quad , \quad a \geq 4. \\ h(a, b, 0, 0) &= b-1 \quad , \quad a \geq 0, \quad b \geq 1. \\ h(0, 0, 1, 0) &= -\frac{1}{2}. \\ h(1, 0, 1, 0) &= *. \\ h(a, 0, 1, 0) &= \{\{\{a-2 | 2\} | 1\} | 0\} \quad , \quad a \geq 2. \\ h(a, b, 1, 0) &= b* \quad , \quad a \geq 0, \quad b \geq 1. \\ h(a, b, c, 0) &= \{\{a+2b+c-2 | b+1\} | b\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \end{aligned}$$

$$\begin{aligned}
h(0, 0, 0, 1) &= -1. \\
h(1, 0, 0, 1) &= 0. \\
h(a, 0, 0, 1) &= \{\{a - 2 \mid 2\} \mid 1\} && , a \geq 2. \\
h(a, b, 0, 1) &= b && , a \geq 0, b \geq 1. \\
h(a, b, c, 1) &= \{a + 2b + c - 1 \mid b + 1\} && , a \geq 0, b \geq 0, c \geq 1. \\
\\
i(0, 0, 0) &= -2. \\
i(1, 0, 0) &= \{0 \mid -\frac{1}{2}\}. \\
i(a, 0, 0) &= \{\{a - 2 \mid \frac{1}{2}\} \mid \{\{\{a - 3 \mid 1\} \mid 0\} \mid 0\}\} && , a \geq 2. \\
i(0, 0, c) &= -1 && , c \geq 1. \\
i(a, 0, c) &= \{\{a - 2 \mid 1\} \mid 0\} && , a \geq 1, c \geq 1. \\
i(a, 1, 0) &= \{a - 1 \mid \frac{1}{2}\} && , a \geq 0. \\
i(a, 1, c) &= \{a - 1 \mid 1\} && , a \geq 0, c \geq 1. \\
i(a, b, c) &= a && , a \geq 0, b \geq 2, c \geq 0. \\
\\
j(0, 0, 0) &= -1. \\
j(1, 0, 0) &= -\frac{1}{2}. \\
j(a, 0, 0) &= \{\{\{a - 3 \mid 1\} \mid 0\} \mid 0\} && , a \geq 2. \\
j(a, b, 0) &= b - \frac{1}{2} && , a \geq 0, b \geq 1. \\
j(a, b, c) &= b && , a \geq 0, b \geq 0, c \geq 1. \\
\\
k(a, b) &= b && , a \geq 0, b \geq 0.
\end{aligned}$$

ClassA32:  $\square\square\square FF$

Let  $f(a, b, c, d, e)$  be the value of  $T^a \square T^b \square T^c \square T^d F T^e F$ .

Let  $g(a, b, c, d, e)$  be the value of  $T^a \square T^b \square T^c F T^d \square T^e F$ .

Let  $h(a, b, c, d, e)$  be the value of  $T^a \square T^b F T^c \square T^d \square T^e F$ .

Let  $i(a, b, c, d, e)$  be the value of  $T^a F T^b \square T^c \square T^d \square T^e F$ .

Let  $j(a, b, c, d)$  be the value of  $T^a \square T^b \square T^c F T^d F \square$ .

Let  $k(a, b, c, d)$  be the value of  $T^a \square T^b F T^c \square T^d F \square$ .

Let  $l(a, b, c, d)$  be the value of  $T^a F T^b \square T^c \square T^d F \square$ .

Let  $m(a, b, c)$  be the value of  $T^a \square T^b F T^c F \square \square$ .

Let  $n(a, b, c)$  be the value of  $T^a F T^b \square T^c F \square \square$ .

Let  $o(a, b)$  be the value of  $T^a F T^b F \square \square \square$ .

**Note:** In  $f, g, h, i$  we omit the case when  $e \geq 2$  since they will reduce to ClassA31.

Values:

**First for  $e = 0$**

$$\begin{aligned}
f(0,0,0,0,0) &= -6. \\
f(1,0,0,0,0) &= (-2) * . \\
f(a,0,0,0,0) &= (a-2)* , \quad a \geq 2. \\
f(0,1,0,0,0) &= -2. \\
f(a,1,0,0,0) &= a-1 , \quad a \geq 1. \\
f(a,2,0,0,0) &= \{2a-1 \mid a\} , \quad a \geq 0. \\
f(a,b,0,0,0) &= \{2a+b-\frac{5}{2} \mid a\} , \quad a \geq 0, \quad b \geq 3. \\
f(0,0,1,0,0) &= -3. \\
f(a,0,1,0,0) &= \{2a-3 \mid a-1\} , \quad a \geq 1. \\
f(a,1,1,0,0) &= 2a-1 , \quad a \geq 0. \\
f(a,b,1,0,0) &= 2a+b-\frac{3}{2} , \quad a \geq 0, \quad b \geq 2. \\
f(0,0,2,0,0) &= (-1) * . \\
f(1,0,2,0,0) &= 1 * . \\
f(a,0,2,0,0) &= 2a-\frac{3}{2} , \quad a \geq 2. \\
f(0,1,2,0,0) &= \frac{1}{2} * . \\
f(a,1,2,0,0) &= 2a , \quad a \geq 1. \\
f(a,b,2,0,0) &= 2a+b-1 , \quad a \geq 0, \quad b \geq 2. \\
f(a,b,c,0,0) &= (2a+b)* , \quad a \geq 0, \quad b \geq 0, \quad c \geq 3.
\end{aligned}$$

$$\begin{aligned}
f(0,0,0,1,0) &= -4. \\
f(1,0,0,1,0) &= -1. \\
f(a,0,0,1,0) &= a-1 , \quad a \geq 2. \\
f(0,1,0,1,0) &= (-1) * . \\
f(1,1,0,1,0) &= 1 * . \\
f(a,1,0,1,0) &= \{\{(2a-2)* \mid a+\frac{1}{2}\} \mid a\} , \quad a \geq 2. \\
f(0,2,0,1,0) &= \frac{1}{2} * . \\
f(a,b,0,1,0) &= \{(2a+b-2)* \mid (a+1)*\} , \quad a \geq 1, \quad b=2 \text{ or } a \geq 0, \quad b \geq 3. \\
f(0,0,1,1,0) &= \{0 \mid (-1)*\}. \\
f(1,0,1,1,0) &= \{3 \mid 1*\}. \\
f(a,0,1,1,0) &= \{3a \mid \{\{(2a-2)* \mid a+\frac{1}{2}\} \mid a\}\} , \quad a \geq 2. \\
f(0,1,1,1,0) &= \{2 \mid \frac{1}{2}\}. \\
f(a,b,1,1,0) &= \{3a+2b \mid \{(2a+b-1)* \mid (a+1)*\}\} , \quad a \geq 1, \quad b=1 \text{ or } a \geq 0, \quad b \geq 2. \\
f(a,b,c,1,0) &= \{3a+2b+c-1 \mid (2a+b)*\} , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \\
f(a,b,c,d,0) &= 3a+2b+c , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2.
\end{aligned}$$

**Second for  $e = 1$**

$$\begin{aligned}
f(0,0,0,0,1) &= -5. \\
f(1,0,0,0,1) &= (-1) * . \\
f(2,0,0,0,1) &= 1 * . \\
f(a,0,0,0,1) &= \{\{(2a-4)* \mid a-\frac{1}{2}\} \mid a-1\} , \quad a \geq 3. \\
f(0,1,0,0,1) &= -1. \\
f(1,1,0,0,1) &= 1. \\
f(a,1,0,0,1) &= \{(2a-2)* \mid a+\frac{1}{2}\} , \quad a \geq 2. \\
f(a,b,0,0,1) &= \{2a+b-2 \mid a+1\} , \quad a \geq 0, \quad b \geq 2.
\end{aligned}$$

$$\begin{aligned}
f(0,0,1,0,1) &= -\frac{3}{2}. \\
f(1,0,1,0,1) &= \frac{1}{2}. \\
f(a,0,1,0,1) &= (2a-2)*, & a \geq 2. \\
f(a,1,1,0,1) &= 2a, & a \geq 0. \\
f(a,b,1,0,1) &= \{ \{2a+b-1 \mid \{2a+b-1 \mid a+2\} \} \mid \{2a+b-1 \mid a+1\} \}, & a \geq 0, b \geq 2. \\
f(a,0,2,0,1) &= (2a)*, & a \geq 0. \\
f(a,b,2,0,1) &= \{ \{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\} \} \} \mid 2a+b \}, & a \geq 0, b \geq 1. \\
f(a,b,c,0,1) &= \{ \{3a+2b+c-2 \mid 2a+b+\frac{1}{2} \} \mid 2a+b \}, & a \geq 0, b \geq 0, c \geq 3.
\end{aligned}$$

$$\begin{aligned}
f(a,0,0,1,1) &= 2a-2, & a \geq 0. \\
f(a,1,0,1,1) &= (2a)*, & a \geq 0. \\
f(a,b,0,1,1) &= \{2a+b-1 \mid \\
&\quad \{ (2a+b-1)*, \{2a+b-1 \mid \{2a+b-1 \mid a+2\} \} \mid \\
&\quad \{2a+b-1 \mid a+1\}, \{ \{2a+b-1 \mid a+2\} \mid a+1 \} \} \}, & a \geq 0, b \geq 2. \\
f(0,0,1,1,1) &= \{0 \mid *\}. \\
f(a,0,1,1,1) &= \{3a \mid \{2a \mid a+1\} \}, & a \geq 1. \\
f(a,b,1,1,1) &= \{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\} \} \}, & a \geq 0, b \geq 1. \\
f(a,b,c,1,1) &= \{3a+2b+c-1 \mid 2a+b+\frac{1}{2} \}, & a \geq 0, b \geq 0, c \geq 2. \\
f(a,b,c,d,1) &= 3a+2b+c, & a \geq 0, b \geq 0, c \geq 0, d \geq 2.
\end{aligned}$$

**For  $e = 0$ , I)  $a = 0$ ,  $b = 0$ ,  $c = 0$**

$$\begin{aligned}
g(0,0,0,0,0) &= -5. \\
g(0,0,0,1,0) &= -3. \\
g(0,0,0,d,0) &= \{ \{d-4 \mid -1\} \mid -2 \}, & d \geq 2.
\end{aligned}$$

**For  $e = 0$ , II)  $d = 0$**

$$\begin{aligned}
g(1,0,0,0,0) &= -2. \\
g(a,0,0,0,0) &= a-2, & a \geq 2. \\
g(0,1,0,0,0) &= (-2)*. \\
g(a,1,0,0,0) &= (a-1)*, & a \geq 1. \\
g(0,2,0,0,0) &= (-\frac{1}{2})*. \\
g(a,2,0,0,0) &= a, & a \geq 1. \\
g(a,b,0,0,0) &= a, & a \geq 0, b \geq 3. \\
g(0,0,1,0,0) &= -2. \\
g(a,0,1,0,0) &= a-1, & a \geq 1. \\
g(a,1,1,0,0) &= \{2a-1 \mid a\}, & a \geq 0. \\
g(a,b,1,0,0) &= \{2a+b-\frac{3}{2} \mid a\}, & a \geq 0, b \geq 2. \\
g(a,0,2,0,0) &= 2a-1, & a \geq 0. \\
g(a,b,2,0,0) &= 2a+b-\frac{1}{2}, & a \geq 0, b \geq 1. \\
g(a,0,3,0,0) &= \{(2a)* \mid 2a\}, & a \geq 0. \\
g(a,b,3,0,0) &= \{ \{ \{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\} \} \} \mid 2a+b \} \mid 2a+b \}, & a \geq 0, b \geq 1. \\
g(a,b,c,0,0) &= \{ \{ \{3a+2b+c-3 \mid 2a+b+\frac{1}{2} \} \mid 2a+b \} \mid 2a+b \}, & a \geq 0, b \geq 0, c \geq 4.
\end{aligned}$$

For  $e = 0$ , III)  $c = 0$

$$\begin{aligned}
g(1, 0, 0, 1, 0) &= -\frac{1}{2}. \\
g(a, 0, 0, 1, 0) &= (a-1)*, & , a \geq 2. \\
g(a, 0, 0, d, 0) &= \{\{2a+d-4 \mid a\} \mid a-1\}, & , a \geq 1, d \geq 2. \\
g(0, 1, 0, 1, 0) &= -1. \\
g(0, 1, 0, d, 0) &= \{\{(d-2)* \mid \frac{1}{2}\} \mid -\frac{1}{2}\}, & , d \geq 2. \\
g(a, 1, 0, 1, 0) &= a, & , a \geq 1. \\
g(a, 1, 0, d, 0) &= \{\{(2a+d-2)* \mid (a+1)*\} \mid a*\}, & , a \geq 1, d \geq 2. \\
g(0, 2, 0, 1, 0) &= \frac{1}{2}. \\
g(a, 2, 0, 1, 0) &= (a+1)*, & , a \geq 1. \\
g(a, b, 0, 1, 0) &= (a+1)*, & , a \geq 0, b \geq 3. \\
g(a, b, 0, 2, 0) &= \{\{(2a+b-1)*, \{2a+b-1 \mid \{2a+b-1 \mid a+2\}\} \mid \\
&\quad \{2a+b-1 \mid a+2\} \mid a+1\}, \{2a+b-1 \mid a+1\}\} \mid \\
&\quad \{\{2a+b-2 \mid a+1\} \mid a\}\}, & , a \geq 0, b \geq 2. \\
g(a, b, 0, d, 0) &= \{\{(2a+b+d-3)* \mid \{\{2a+b-1 \mid a+2\} \mid a+1\}\} \mid \\
&\quad \{\{2a+b-2 \mid a+1\} \mid a\}\}, & , a \geq 0, b \geq 2, d \geq 3.
\end{aligned}$$

For  $e = 0$ , IV)  $c = 1$

$$\begin{aligned}
g(0, 0, 1, 1, 0) &= (-1)*. \\
g(1, 0, 1, 1, 0) &= 1*. \\
g(a, 0, 1, 1, 0) &= \{\{(2a-2)* \mid a+\frac{1}{2}\} \mid a\}, & , a \geq 2. \\
g(0, 1, 1, 1, 0) &= \frac{1}{2}*. \\
g(a, b, 1, 1, 0) &= \{(2a+b-1)* \mid (a+1)*\}, & , a \geq 1, b=1 \text{ or } a \geq 0, b \geq 2. \\
g(0, 0, 1, d, 0) &= \{\{d-2 \mid 0\} \mid -1\}, & , d \geq 2. \\
g(a, 0, 1, d, 0) &= \{\{2a+d-2 \mid a+1\} \mid a\}, & , a \geq 1, d \geq 2. \\
g(a, b, 1, d, 0) &= \{\{2a+b+d-2 \mid \{2a+b \mid a+2\}\} \mid \\
&\quad \{2a+b-1 \mid a+1\}, a+2\}, & , a \geq 0, b \geq 1, d \geq 2.
\end{aligned}$$

For  $e = 0$ , V)  $c \geq 1$

$$\begin{aligned}
g(a, b, c, 1, 0) &= (2a+b)*, & , a \geq 0, b \geq 0, c \geq 2. \\
g(a, b, c, d, 0) &= \{\{2a+b+d-2 \mid 2a+b+1\} \mid 2a+b\}, & , a \geq 0, b \geq 0, c \geq 2, d \geq 2.
\end{aligned}$$

For  $e = 1$ , I)  $a = 0, b = 0, c = 0$

$$\begin{aligned}
g(0, 0, 0, 0, 1) &= -4. \\
g(0, 0, 0, d, 1) &= \{d-3 \mid -1\}, & , d \geq 1.
\end{aligned}$$

**For  $e = 1$ , II)  $d = 0$**

$$\begin{aligned}
g(1, 0, 0, 0, 1) &= -1. \\
g(a, 0, 0, 0, 1) &= a - 1 && , a \geq 2. \\
g(0, 1, 0, 0, 1) &= (-1) * . \\
g(1, 1, 0, 0, 1) &= 1 * . \\
g(a, 1, 0, 0, 1) &= \{(2a - 2) * | a + \frac{1}{2} | a\} && , a \geq 2. \\
g(0, 2, 0, 0, 1) &= (\frac{1}{2}) * . \\
g(a, 2, 0, 0, 1) &= a + 1 && , a \geq 1. \\
g(a, b, 0, 0, 1) &= a + 1 && , a \geq 0, b \geq 3. \\
g(0, 0, 1, 0, 1) &= -1. \\
g(1, 0, 1, 0, 1) &= 1. \\
g(a, 0, 1, 0, 1) &= \{(2a - 2) * | a + \frac{1}{2}\} && , a \geq 2. \\
g(a, b, 1, 0, 1) &= \{2a + b - 1 | a + 1\} && , a \geq 0, b \geq 1. \\
g(a, b, c, 0, 1) &= 2a + b && , a \geq 0, b \geq 0, c \geq 2.
\end{aligned}$$

**For  $e = 1$ , III)  $c = 0$**

$$\begin{aligned}
g(a, 0, 0, d, 1) &= \{2a + d - 3 | a\} && , a \geq 1, d \geq 1. \\
g(0, 1, 0, d, 1) &= \{(d - 1) * | \frac{1}{2}\} && , d \geq 1. \\
g(a, 1, 0, d, 1) &= \{(2a + d - 1) * | (a + 1) * \} && , a \geq 1, d \geq 1. \\
g(a, b, 0, 1, 1) &= \{(2a + b - 1) *, \{2a + b - 1 | \{2a + b - 1 | a + 2\}\} | \\
&\quad \{2a + b - 1 | a + 1\}, \{\{2a + b - 1 | a + 2\} | a + 1\}\} && , a \geq 0, b \geq 2. \\
g(a, b, 0, d, 1) &= \{(2a + b + d - 2) * | \{\{2a + b - 1 | a + 2\} | a + 1\}\} && , a \geq 0, b \geq 2, d \geq 2.
\end{aligned}$$

**For  $e = 1$ , IV)  $c = 1$**

$$\begin{aligned}
g(0, 0, 1, d, 1) &= \{d - 1 | 0\} && , d \geq 1. \\
g(a, 0, 1, d, 1) &= \{2a + d - 1 | a + 1\} && , a \geq 1, d \geq 1. \\
g(a, b, 1, d, 1) &= \{2a + b + d - 1 | \{2a + b | a + 2\}\} && , a \geq 0, b \geq 1, d \geq 1.
\end{aligned}$$

**For  $e = 1$ , V)  $c \geq 2$**

$$g(a, b, c, d, 1) = \{2a + b + d - 1 | 2a + b + 1\} \quad , a \geq 0, b \geq 0, c \geq 2, d \geq 1.$$

**For  $e = 0$ , I)  $a = 0, b = 0$**

$$\begin{aligned}
h(0, 0, 0, 0, 0) &= -4. \\
h(0, 0, c, 0, 0) &= c - 2 && , c \geq 1. \\
h(0, 0, 0, 1, 0) &= -2. \\
h(0, 0, c, 1, 0) &= (c - 1) * && , c \geq 1. \\
h(0, 0, c, d, 0) &= \{\{2c + d - 3 | c\} | c - 1\} && , c \geq 0, d \geq 2.
\end{aligned}$$

**For  $e = 0$ , II)  $c = 0$ ,  $d = 0$**

$$\begin{aligned}
h(1, 0, 0, 0, 0) &= -1. \\
h(a, 0, 0, 0, 0) &= \{a - 2 \mid \{a - 2 \mid 3\}\} \quad , \quad a \geq 2. \\
h(0, 1, 0, 0, 0) &= -2. \\
h(a, 1, 0, 0, 0) &= a - 1 \quad , \quad a \geq 1. \\
h(0, 2, 0, 0, 0) &= -\frac{1}{2}. \\
h(a, 2, 0, 0, 0) &= a^* \quad , \quad a \geq 1. \\
h(0, 3, 0, 0, 0) &= \{\frac{1}{2} \mid 0\}. \\
h(a, 3, 0, 0, 0) &= \{(a + 1)^* \mid a\} \quad , \quad a \geq 1. \\
h(a, b, 0, 0, 0) &= \{(a + 1)^* \mid a\} \quad , \quad a \geq 0, \quad b \geq 4.
\end{aligned}$$

**For  $e = 0$ , III)  $b = 0$**

$$\begin{aligned}
h(a, 0, c, 0, 0) &= a + c - \frac{3}{2} \quad , \quad a \geq 1, \quad c \geq 1. \\
h(1, 0, 0, 1, 0) &= *. \\
h(2, 0, 0, 1, 0) &= \frac{3}{2}*. \\
h(3, 0, 0, 1, 0) &= \frac{5}{2}*. \\
h(4, 0, 0, 1, 0) &= \frac{27}{8}. \\
h(a, 0, 0, 1, 0) &= \{a - 1 \mid \{a - 1 \mid 5\}\} \quad , \quad a \geq 5. \\
h(a, 0, c, 1, 0) &= (a + c - \frac{1}{2})^* \quad , \quad a \geq 1, \quad c \geq 1. \\
h(a, 0, 0, 2, 0) &= \{\{2a - 1 \mid a + \frac{1}{2}\} \mid a - \frac{1}{2}\} \quad , \quad a = 1, 2, 3, 4, 5. \\
h(6, 0, 0, 2, 0) &= \{\{11 \mid 6^*\} \mid \frac{11}{2}\}. \\
h(a, 0, 0, 2, 0) &= \{a \mid a - \frac{1}{2}\} \quad , \quad a \geq 7. \\
h(a, 0, c, d, 0) &= \{\{2a + 2c + d - 3 \mid a + c + \frac{1}{2}\} \mid a + c - \frac{1}{2}\} \quad , \quad a \geq 1, \quad c \geq 1, \quad d = 2 \\
&\quad \text{or } a \geq 1, \quad c \geq 0, \quad d \geq 3.
\end{aligned}$$

**For  $e = 0$ , IV)  $d = 0, 1$ ,  $b \geq 1$**

$$\begin{aligned}
h(a, b, c, 0, 0) &= a + c - 1 \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 1. \\
h(0, 1, 0, 1, 0) &= -\frac{1}{2}. \\
h(a, 1, 0, 1, 0) &= a^* \quad , \quad a \geq 1. \\
h(0, 2, 0, 1, 0) &= \{\frac{1}{2} \mid 0\}. \\
h(a, 2, 0, 1, 0) &= \{(a + 1)^* \mid a\} \quad , \quad a \geq 1. \\
h(a, b, 0, 1, 0) &= \{\{\{2a + b - 2^*, \{2a + b - 2 \mid \\
&\quad \{2a + b - 2 \mid a + 2\}\} \mid \{2a + b - 2 \mid a + 1\}, \\
&\quad \{2a + b - 2 \mid a + 2\} \mid a + 1\}\} \mid a + 1\} \mid a\} \\
&\quad , \quad a \geq 0, \quad b \geq 3. \\
h(a, b, c, 1, 0) &= (a + c)^* \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 1.
\end{aligned}$$

**For  $e = 0$ , V)  $d \geq 2$ ,  $b \geq 1$**

$$h(a, b, c, d, 0) = \{\{2a + b + 2c + d - 3 \mid a + c + 1\} \mid a + c\} \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 0, \quad d \geq 2.$$

**For**  $e = 1$ , **I)**  $a = 0, b = 0$

$$\begin{aligned} h(0, 0, 0, 0, 1) &= -3. \\ h(0, 0, c, 0, 1) &= c - 1, \quad c \geq 1. \\ h(0, 0, c, d, 1) &= \{2c + d - 2 \mid c\}, \quad c \geq 0, d \geq 1. \end{aligned}$$

**For**  $e = 1$ , **II)**  $c = 0, d = 0$

$$\begin{aligned} h(1, 0, 0, 0, 1) &= 0. \\ h(a, 0, 0, 0, 1) &= \{a - 1 \mid \{a - 1 \mid 4\}\}, \quad a \geq 2. \\ h(0, 1, 0, 0, 1) &= -1. \\ h(a, 1, 0, 0, 1) &= a, \quad a \geq 1. \\ h(0, 2, 0, 0, 1) &= \frac{1}{2}. \\ h(a, 2, 0, 0, 1) &= (a + 1)*, \quad a \geq 1. \\ h(a, b, 0, 0, 1) &= \{\{2a + b - 2*, \{2a + b - 2 \mid \{2a + b - 2 \mid a + 2\}\} \\ &\quad \mid \{\{2a + b - 2 \mid a + 2\} \mid a + 1\}, \\ &\quad \{2a + b - 2 \mid a + 1\}\} \mid a + 1\} \\ &\quad, \quad a \geq 0, b \geq 3. \end{aligned}$$

**For**  $e = 1$ , **III)**  $b = 0$

$$\begin{aligned} h(a, 0, c, 0, 1) &= a + c - \frac{1}{2}, \quad a \geq 1, c \geq 1. \\ h(a, 0, 0, 1, 1) &= \{2a - 1 \mid a + \frac{1}{2}\}, \quad 1 \leq a \leq 4. \\ h(a, 0, 0, 1, 1) &= \{2a - 1 \mid \{a \mid 6\}\}, \quad a \geq 5. \\ h(a, 0, c, 1, 1) &= \{2a + 2c - 1 \mid a + c + \frac{1}{2}\}, \quad a \geq 1, c \geq 1. \\ h(a, 0, c, d, 1) &= \{2a + 2c + d - 2 \mid a + c + \frac{1}{2}\}, \quad a \geq 1, c \geq 0, d \geq 2 \end{aligned}$$

**For**  $e = 1$ , **IV)**  $d = 0, b \geq 1$

$$h(a, b, c, 0, 1) = a + c, \quad a \geq 0, b \geq 1, c \geq 1.$$

**For**  $e = 1$ , **V)**  $d \geq 1, b \geq 1$

$$h(a, b, c, d, 1) = \{2a + b + 2c + d - 2 \mid a + c + 1\}, \quad a \geq 0, b \geq 1, c \geq 0, d \geq 1.$$



**First for  $d = 0$ ,  $e = 0$**

$$\begin{aligned}
i(0, 0, 0, 0, 0) &= -3. \\
i(1, 0, 0, 0, 0) &= (-1)*. \\
i(a, 0, 0, 0, 0) &= a - 1 && , a \geq 2. \\
i(a, 1, 0, 0, 0) &= a* && , a \geq 0. \\
i(a, b, 0, 0, 0) &= \{a + 2b - 2 \mid a + 2b - 2\} && , a = 0, 1, 2, b \geq 2. \\
i(3, b, 0, 0, 0) &= (2b + \frac{1}{2})* && , b \geq 2. \\
i(a, b, 0, 0, 0) &= a + 2b - 3 && , a \geq 4, b \geq 2. \\
\\
i(a, b, c, 0, 0) &= a + 2b + c - 1 && , a = 0, 1, 2, b \geq 0, c \geq 1. \\
i(3, 0, 1, 0, 0) &= 3. \\
i(3, b, 1, 0, 0) &= 2b + \frac{5}{2} && , b \geq 1. \\
i(3, b, c, 0, 0) &= 2b + c + 2 && , b \geq 0, c \geq 2. \\
i(a, 0, 1, 0, 0) &= a && , a \geq 4. \\
i(a, b, 1, 0, 0) &= (a + 2b - 1)* && , a \geq 4, b \geq 1. \\
i(a, b, c, 0, 0) &= a + 2b + c - 1 && , a \geq 0, b \geq 0, c \geq 2.
\end{aligned}$$

**Second for  $d \geq 1$ ,  $e = 0$**

$$\begin{aligned}
i(0, 0, 0, 1, 0) &= -1. \\
i(1, 0, 0, 1, 0) &= 0. \\
i(a, 0, 0, 1, 0) &= a* && , a \geq 2. \\
i(a, b, 0, 1, 0) &= a + 2b - 1 && , a \geq 0, b \geq 1. \\
\\
i(a, b, 1, 1, 0) &= (a + 2b + 1)* && , a = 0, 1, 2, b \geq 0. \\
i(3, 0, 1, 1, 0) &= 4*. \\
i(3, b, 1, 1, 0) &= 2b + \frac{15}{4} && , b \geq 1. \\
i(a, 0, 1, 1, 0) &= (a + 1)* && , a \geq 4. \\
i(a, b, 1, 1, 0) &= a + 2b && , a \geq 4, b \geq 1. \\
i(a, b, c, 1, 0) &= (a + 2b + c)* && , a \geq 0, b \geq 0, c \geq 2. \\
\\
i(a, b, c, 2, 0) &= \{\{2a + 3b + 2c \mid a + 2b + c + 1\} \mid a + 2b + c\} && , a = 0, 1, 2, b \geq 0, c \geq 0. \\
i(3, 0, 0, 2, 0) &= \{\{6 \mid 4\} \mid 3\}. \\
i(3, b, 0, 2, 0) &= \{\{3b + 6 \mid 2b + \frac{7}{2}\} \mid 2b + 3\} && , b \geq 1. \\
i(3, b, c, 2, 0) &= \{\{3b + 2c + 6 \mid 2b + c + 4\} \mid 2b + c + 3\} && , b \geq 0, c \geq 1. \\
i(a, 0, 0, 2, 0) &= \{\{2a \mid a + 1\} \mid a\} && , a \geq 4. \\
i(a, b, 0, 2, 0) &= (a + 2b)* && , a \geq 4, b \geq 1. \\
i(a, b, c, 2, 0) &= \{\{2a + 3b + 2c \mid a + 2b + c + 1\} \mid a + 2b + c\} && , a \geq 4, b \geq 0, c \geq 1. \\
i(a, b, c, d, 0) &= \{\{2a + 3b + 2c + d - 2 \mid a + 2b + c + 1\} \mid a + 2b + c\} && , a \geq 0, b \geq 0, c \geq 0, d \geq 3.
\end{aligned}$$

**Third for**  $d = 0, e = 1$

$$\begin{aligned}
i(0, 0, 0, 0, 1) &= -2. \\
i(1, 0, 0, 0, 1) &= *. \\
i(a, 0, 0, 0, 1) &= a, \quad a \geq 2. \\
i(a, 1, 0, 0, 1) &= (a+1)*, \quad a \geq 0. \\
\\
i(a, b, 0, 0, 1) &= \{a+2b-1 \mid a+2b-1\}, \quad a = 0, 1, 2, \quad b \geq 2. \\
i(3, b, 0, 0, 1) &= (2b + \frac{3}{2})*, \quad b \geq 2. \\
i(a, b, 0, 0, 1) &= a+2b-2, \quad a \geq 4, \quad b \geq 2. \\
\\
i(a, b, c, 0, 1) &= a+2b+c, \quad a = 0, 1, 2, \quad b \geq 0, \quad c \geq 1. \\
i(3, 0, 1, 0, 1) &= 4. \\
i(3, b, 1, 0, 1) &= 2b + \frac{7}{2}, \quad b \geq 1. \\
i(3, b, c, 0, 1) &= 2b+c+3, \quad b \geq 0, \quad c \geq 2. \\
i(a, 0, 1, 0, 1) &= a+1, \quad a \geq 4. \\
i(a, b, 1, 0, 1) &= (a+2b)*, \quad a \geq 4, \quad b \geq 1. \\
i(a, b, c, 0, 1) &= a+2b+c, \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.
\end{aligned}$$

**Note:**  $i(a, b, c, 0, 1) = i(a, b, c, 0, 0) + 1, \quad a \geq 0, \quad b \geq 0, \quad c \geq 0.$

**Fourth for**  $d \geq 1, e = 1$

$$\begin{aligned}
i(a, 0, 0, 1, 1) &= \{2a \mid a+1\}, \quad a \geq 0. \\
i(a, b, 0, 1, 1) &= \{2a+3b \mid \{a+2b \mid 2b+4\}\}, \quad a \geq 0, \quad b \geq 1. \\
i(a, b, c, 1, 1) &= \{2a+3b+2c \mid a+2b+c+1\}, \quad a \geq 0, \quad b \geq 0, \quad c \geq 1. \\
i(a, b, c, d, 1) &= \{2a+3b+2c+d-1 \mid a+2b+c+1\}, \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2.
\end{aligned}$$

**First for**  $c = 0, d = 0$

$$\begin{aligned}
j(0, 0, 0, 0) &= -4. \\
j(1, 0, 0, 0) &= -1. \\
j(a, 0, 0, 0) &= \{a-2 \mid \{a-2 \mid 3\}\}, \quad a \geq 2. \\
j(0, 1, 0, 0) &= (-1)*. \\
j(a, 1, 0, 0) &= \{a-\frac{1}{2} \mid \{a-1 \mid \{a-1 \mid 3\}\}\}, \quad a \geq 1. \\
j(0, 2, 0, 0) &= \{\frac{1}{4} \mid -\frac{1}{4}\}. \\
j(a, 2, 0, 0) &= \{\{2a \mid a+\frac{1}{2}\} \mid \{a* \mid a\}\}, \quad a \geq 1. \\
j(a, b, 0, 0) &= \{\{2a+b-2 \mid a+\frac{1}{2}\} \mid \{\{2a+b-3 \mid a+1\} \mid a\} \mid a\}\}, \quad a \geq 0, \quad b \geq 3.
\end{aligned}$$

**Second for**  $c = 0, d \geq 1$

$$\begin{aligned}
j(0, 0, 0, d) &= -2, \quad d \geq 1. \\
j(a, 0, 0, d) &= a-1, \quad a \geq 1, \quad d \geq 1. \\
j(0, 1, 0, d) &= -\frac{1}{2}, \quad d \geq 1. \\
j(a, 1, 0, d) &= a*, \quad a \geq 1, \quad d \geq 1. \\
j(a, b, 0, d) &= \{\{2a+b-2 \mid a+1\} \mid a\}, \quad a \geq 0, \quad b \geq 2, \quad d \geq 1.
\end{aligned}$$

**Third for  $c = 1, d = 0$**

$$\begin{aligned} j(0, 0, 1, 0) &= -1. \\ j(a, 0, 1, 0) &= a - \frac{1}{2}, \quad a \geq 1. \\ j(a, b, 1, 0) &= \{2a + b - 1 \mid a + \frac{1}{2}\}, \quad a \geq 0, b \geq 1. \end{aligned}$$

**Fourth for  $c = 1, d \geq 1$**

$$\begin{aligned} j(0, 0, 1, d) &= -1, \quad d \geq 1. \\ j(a, 0, 1, d) &= a, \quad a \geq 1, d \geq 1. \\ j(a, b, 1, d) &= \{2a + b - 1 \mid a + 1\}, \quad a \geq 0, b \geq 1, d \geq 1. \end{aligned}$$

**Fifth for  $c \geq 2$**

$$j(a, b, c, d) = 2a + b, \quad a \geq 0, b \geq 3, c \geq 2, d \geq 0.$$

**First for  $b = 0, d = 0$**

$$\begin{aligned} k(0, 0, 0, 0) &= -3. \\ k(1, 0, 0, 0) &= (-1)*. \\ k(a, 0, 0, 0) &= \{a - 2 \mid 3\}, \quad a \geq 2. \\ k(0, 0, c, 0) &= (c - 2)*, \quad c \geq 1. \\ k(a, 0, c, 0) &= a + c - 1, \quad a \geq 1, c \geq 1. \end{aligned}$$

**Second for  $b \geq 1, d = 0$**

$$\begin{aligned} k(0, 1, 0, 0) &= -1. \\ k(a, 1, 0, 0) &= \{a - 1 \mid \{a - 1 \mid 3\}\}, \quad a \geq 1. \\ k(0, 2, 0, 0) &= -\frac{1}{4}. \\ k(a, 2, 0, 0) &= \{a* \mid a\}, \quad a \geq 1. \\ k(a, b, 0, 0) &= \{\{2a + b - 3 \mid a + 1\} \mid a\}, \quad a \geq 0, b \geq 3. \\ k(a, b, c, 0) &= a + c - \frac{1}{2}, \quad a \geq 0, b \geq 1, c \geq 1. \end{aligned}$$

**Third for  $b = 0, d \geq 1$**

$$\begin{aligned} k(0, 0, c, d) &= c - 1, \quad c \geq 0, d \geq 1. \\ k(a, 0, 0, 1) &= a - \frac{1}{2}, \quad a = 1, 2, 3, 4. \\ k(a, 0, 0, 1) &= \{a - 1 \mid 5\}, \quad a \geq 5. \\ k(a, 0, c, d) &= a + c - \frac{1}{2}, \quad a \geq 1, c \geq 1, d = 1 \\ &\text{or } a \geq 1, c \geq 0, d \geq 2. \end{aligned}$$

**Fourth for  $b \geq 1, d \geq 1$**

$$k(a, b, c, d) = a + c, \quad a \geq 0, \quad b \geq 1, \quad c \geq 0, \quad d \geq 1.$$

$$\begin{aligned} l(0, 0, 0, 0) &= -2. \\ l(1, 0, 0, 0) &= -1. \\ l(a, 0, 0, 0) &= (a - 1)*, & a \geq 2. \\ l(a, 1, 0, 0) &= \{a* \mid 3\}, & a \geq 0. \\ l(a, b, 0, 0) &= \{a + 2b - 3 \mid 2b + 1\}, & a \geq 0, \quad b \geq 2. \\ l(a, 0, 1, 0) &= a*, & a \geq 0. \\ \\ l(a, b, 1, 0) &= (a + 2b)*, & a = 0, 1, 2, \quad b \geq 1. \\ l(3, b, 1, 0) &= (2b + \frac{5}{2})*, & b \geq 1. \\ l(a, b, 1, 0) &= a + 2b - 1, & a \geq 4, \quad b \geq 1. \\ \\ l(a, b, c, 0) &= (a + 2b + c - 1)*, & a \geq 0, \quad b \geq 0, \quad c \geq 2. \\ l(a, 0, 0, 1) &= a, & a \geq 0. \\ l(a, b, 0, 1) &= \{a + 2b - 1 \mid 2b + 3\}, & a \geq 0, \quad b \geq 1. \\ l(a, b, c, d) &= a + 2b + c, & a \geq 0, \quad b \geq 0, \quad c \geq 1, \quad d = 1 \\ & & \text{or } a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2. \end{aligned}$$

**First for  $b = 0, a = 0$**

$$m(0, 0, c) = c - 2, \quad c \geq 0.$$

**Second for  $b = 0, c = 0, (a \geq 1)$**

$$\begin{aligned} m(1, 0, 0) &= \{0 \mid -\frac{1}{2}\}. \\ m(2, 0, 0) &= 1*. \\ m(3, 0, 0) &= 2*. \\ m(4, 0, 0) &= \frac{11}{4}. \\ m(a, 0, 0) &= 3, & a \geq 5. \\ \\ m(a, 0, c) &= (a + c - 1)*, & a \geq 1, \quad c \geq 1. \\ m(a, 1, 0) &= \{a - 1 \mid 3\}, & a \geq 0. \\ m(a, 1, c) &= a + c, & a \geq 0, \quad c \geq 1. \\ m(a, b, c) &= a + c, & a \geq 0, \quad b \geq 2, \quad c \geq 0. \\ \\ n(0, 0, 0) &= -1. \\ n(1, 0, 0) &= -\frac{1}{2}. \\ n(a, 0, 0) &= \{(a - 1)* \mid 0\}, & a \geq 2. \\ n(a, 1, 0) &= \{\{a* \mid 3\} \mid 2\}, & a \geq 0. \\ n(a, b, 0) &= \{\{a + 2b - 3 \mid 2b + 1\} \mid 2b\}, & a \geq 0, \quad b \geq 2. \end{aligned}$$

$$\begin{aligned}
n(a, 0, 1) &= \{a * | 3\} & , & a \geq 0. \\
n(a, b, 1) &= \{a + 2b - 1 | 2b + 3\} & , & a \geq 0, b \geq 1. \\
n(a, b, c) &= a + 2b + c - 1 & , & a \geq 0, b \geq 0, c \geq 2. \\
o(a, b) &= 2b & , & a \geq 0, b \geq 0.
\end{aligned}$$

## 4. RESULT OF CLASS WITH THREE FROGS.

ClassA13:  $\square FFF$ Let  $f(a, b, c, d)$  be the value of  $T^a \square T^b FT^c FT^d F$ .Let  $g(a, b, c, d)$  be the value of  $T^a FT^b \square T^c FT^d F$ .Let  $h(a, b, c, d)$  be the value of  $T^a FT^b FT^c \square T^d F$ .Let  $i(a, b, c)$  be the value of  $T^a FT^b FT^c F \square$ .**Note:**

1) In  $f$  we omit the case where  $c \geq 2$  or  $d \geq 2$  since it will reduce to the results in ClassA11 and ClassA12 respectively.

2) In  $g, h$  we omit the case where  $d \geq 2$  since it will reduce to the results of ClassA12.

$$\begin{aligned}
f(0, 0, 0, 0) &= -3. \\
f(1, 0, 0, 0) &= \{0 | (-1) * \}. \\
f(a, 0, 0, 0) &= * & , & a \geq 2. \\
f(0, 1, 0, 0) &= 0. \\
f(1, 1, 0, 0) &= \{0 | \{0 | -\frac{1}{4}\} \}. \\
f(a, 1, 0, 0) &= \{a - 1 | \{0 | \{ \{ \{ a - 3 | \frac{1}{4} \} | 0 \} | 0 \} | 0 \} \} \} & , & a \geq 2. \\
f(a, b, 0, 0) &= a & , & a \geq 0, b \geq 2. \\
f(0, 0, 1, 0) &= -1. \\
f(a, 0, 1, 0) &= \{ \{ a - 2 | \{0 | * \} \} | 0 \} & , & a \geq 1. \\
f(a, 1, 1, 0) &= \{ a - 1 | \{0 | * \} \} & , & a \geq 0. \\
f(a, b, 1, 0) &= a & , & a \geq 0, b \geq 2. \\
f(0, 0, 0, 1) &= -2. \\
f(1, 0, 0, 1) &= \{0 | -\frac{1}{2}\}. \\
f(a, 0, 0, 1) &= * & , & a \geq 2. \\
f(a, 1, 0, 1) &= \{a - 1 | * \} & , & a \geq 0. \\
f(a, b, 0, 1) &= a & , & a \geq 0, b \geq 2. \\
f(0, 0, 1, 1) &= -1. \\
f(a, 0, 1, 1) &= \{ \{ a - 2 | \frac{1}{4} \} | 0 \} & , & a \geq 1. \\
f(a, 1, 1, 1) &= \{ a - 1 | \frac{1}{4} \} & , & a \geq 0. \\
f(a, b, 1, 1) &= a & , & a \geq 0, b \geq 2.
\end{aligned}$$

$$\begin{aligned}
g(0,0,0,0) &= -2. \\
g(1,0,0,0) &= (-1)*. \\
g(a,0,0,0) &= \{\{\{a-3 \mid \{0 \mid *\}\} \mid 0\} \mid -1\} \quad , \quad a \geq 2. \\
g(0,1,0,0) &= \{0 \mid -\frac{1}{2}\}. \\
g(1,1,0,0) &= \{0 \mid -\frac{1}{4}\}. \\
g(a,1,0,0) &= \{0 \mid \{\{\{a-3 \mid \frac{1}{4}\} \mid 0\} \mid 0\} \mid 0\}\} \quad , \quad a \geq 2. \\
g(a,b,0,0) &= * \quad , \quad a \geq 0, \quad b \geq 2. \\
g(a,b,1,0) &= \{b-1 \mid *\} \quad , \quad a \geq 0, \quad b \geq 0. \\
\\
g(0,0,0,1) &= -1. \\
g(1,0,0,1) &= -\frac{1}{2}. \\
g(a,0,0,1) &= \{\{\{a-3 \mid \frac{1}{4}\} \mid 0\} \mid 0\} \quad , \quad a \geq 2. \\
g(a,b,0,1) &= \{\{b-2 \mid \frac{1}{2}\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 1. \\
g(a,b,1,1) &= \{b-1 \mid \frac{1}{2}\} \quad , \quad a \geq 0, \quad b \geq 0. \\
g(a,b,c,d) &= b \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2, \quad d \geq 0. \\
\\
h(a,0,0,0) &= -1 \quad , \quad a \geq 0. \\
h(a,0,c,0) &= \{\{c-2 \mid 1\} \mid 0\} \quad , \quad a \geq 0, \quad c \geq 1. \\
h(0,1,0,0) &= -\frac{1}{2}. \\
h(1,1,0,0) &= -\frac{1}{4}. \\
h(a,1,0,0) &= \{\{\{\{a-3 \mid \frac{1}{4}\} \mid 0\} \mid 0\} \mid 0\} \quad , \quad a \geq 2. \\
h(a,b,0,0) &= \{\{\{b-3 \mid \frac{1}{2}\} \mid 0\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 2. \\
h(a,b,c,0) &= \{\{c-2 \mid 1\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 1. \\
\\
h(a,0,c,1) &= \{c-1 \mid 1\} \quad , \quad a \geq 0, \quad c \geq 0. \\
h(a,b,0,1) &= 0 \quad , \quad a \geq 0, \quad b \geq 1. \\
h(a,b,1,1) &= \frac{1}{2} \quad , \quad a \geq 0, \quad b \geq 1. \\
h(a,b,c,1) &= \{c-1 \mid 1\} \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 2. \\
\\
i(a,b,c) &= 0 \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0.
\end{aligned}$$

## 5. RESULT OF CLASS B.

ClassB11: TF

Let  $f(a,b,c)$  be the value of  $\square^a T \square^b F \square^c$ .

Values:

$$\begin{aligned}
f(a,b,c) &= \{c-a-2 \mid c-a+2\} \quad , \quad a \geq 0, \quad c \geq 0 \quad \text{and } b \text{ is even.} \\
f(a,b,c) &= \{\{c-a-3 \mid c-a+1\} \mid \{c-a-1 \mid c-a+3\}\} \quad , \quad a \geq 0, \quad c \geq 0 \quad \text{and } b \text{ is odd.}
\end{aligned}$$

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