The Distribution of Double Deficiencies in Pattern-Avoiding Permutations

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Abstract

A double deficiency (DD) of a permutation π of length n is an index k, where $1 \leq k \leq n$, satisfying $\pi(k) < k < \pi^{-1}(k)$. In this talk, we investigate the distribution of the number of **double deficiencies** in length-3 pattern-avoiding permutations.

1 Short Introduction

During the month of June, we got to visit Dr.Z and look at the paper:

Martin Rubey and Christian Stump, *Double Deficiencies of Dyck Paths via the Billey-Jockusch-Stanley Bijection*

together. This talk is the product of our collaboration during the visit.

Pattern avoidance of permutations

A permutation avoids a pattern if no sub-sequence inside it matches the relative size order of that pattern. For example, the permutation $\pi = 2413$ avoids the pattern 321 because no three numbers decrease in the same relative way, but it contains the pattern 132 because the sub-sequence $2, 4, 3$.

Remark: the number of permutations of length $n, \pi(n)$, that avoid a pattern of length 3 is a Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

i.e. $1, 2, 5, 14, 42, 132, 429, 1430, \ldots$.

Refinements

Several refinements of length-3 pattern avoidance have been studied.

- Elizalde determined the joint distribution of **fixed points** and **excedances** for permutations simultaneously avoiding subsets of S_3 [2], while Elizalde and Pak gave a bijection between 321- and 132-avoiding permutations that preserves both statistics [4].

- Dokos, Dwyer, Johnson, Sagan, and Selsor studied this phenomenon more generally through *statistic-Wilf equivalence*, and in particular determined the equivalences arising from the **inversion number** and **major index** for sets of forbidden patterns contained in S_3, [1].

- In this talk, the refining statistic is the number of **double deficiencies**. We want to investigate the distribution of number of double deficiencies, $DD(\pi)$, in each of the pattern-avoidance class.

Examples on double deficiency

Definition. For $\pi \in \mathfrak{S}_n$, an index $k \in [n]$ is a double deficiency of π if

$$\pi(k) < k < \pi^{-1}(k).$$

We denote the set of double deficiencies of π by $\mathcal{D}(\pi)$ and denote the number of double deficiencies of π by $DD(\pi)$.

Example 1.

i	1	2	3
$\pi(i)$	3	1	2

$$\mathcal{D}(\pi) = \mathcal{D}([3, 1, 2]) = \{2\},$$

since $\pi(2) = 1 < 2 < 3 = \pi^{-1}(2)$.

Example 2.

i	1	2	3	4	5	6	7	8
$\pi(i)$	5	6	1	2	3	4	7	8

$$\mathcal{D}(\pi) = \mathcal{D}([5, 6, 1, 2, 3, 4, 7, 8]) = \{3, 4\},$$

since $\pi(3) = 1 < 3 < 5 = \pi^{-1}(3)$ and $\pi(4) = 2 < 4 < 6 = \pi^{-1}(4)$.

Examples on pattern avoidance and generating function of DD

321-pattern avoidance of permutation of length 4 and its DD

π	$\mathcal{D}(\pi)$	π	$\mathcal{D}(\pi)$
$[1,2,3,4]$	$\{\}$	$[2,3,1,4]$	$\{\}$
$[1,2,4,3]$	$\{\}$	$[2,3,4,1]$	$\{\}$
$[1,3,2,4]$	$\{\}$	$[2,4,1,3]$	$\{3\}$
$[1,3,4,2]$	$\{\}$	$[3,1,2,4]$	$\{2\}$
$[1,4,2,3]$	$\{3\}$	$[3,1,4,2]$	$\{2\}$
$[2,1,3,4]$	$\{\}$	$[3,4,1,2]$	$\{\}$
$[2,1,4,3]$	$\{\}$	$[4,1,2,3]$	$\{2,3\}$

Generating function:

$$f_{321}(4,x) = \sum_{\pi \in \mathrm{Av}_4(321)} x^{DD(\pi)} = x^2 + 4x + 9.$$

Goal: To find a quick way to compute the generating function of permutations that avoid pattern of length 3.

2 Generating function of DD on permutations of length n (no restriction)

- There is a bijective map between a one-line notation of n-permutation and n-permutation with cycle notation. For example, $(2)(41)(7635) = 2417635$ with inverse map defined by left-to-right maxima.
- In this section, we consider of a permutation from its the cycle notation.
- In term of cycle notation, DD occurs when there are three consecutive decreasing elements. That is

$$\pi(k-1) > \pi(k) > \pi(k+1).$$

Hence we are considering the generating function of **double descents** of n-permutation.

Known Result

Elizalde and Noy obtained the grand (exponential) generating function for the number of double descents [3]. Consequently, their result also gives the unrestricted double-deficiency distribution :

$$\sum_{n\geq 0} f_{\varnothing}(n,x)\frac{z^n}{n!} = \frac{2r\exp\left(\frac{1-x+r}{2}z\right)}{1+x+r-(1+x-r)\exp(rz)}, \qquad r=\sqrt{(x-1)(x+3)}.$$

Our recurrences for generating functions

Let $U(l, n, x)$ be generating function of $\pi \in S_n$ where $\pi(1) = l$ and $\pi(2) > l$. Similarly, we let $D(l, n, x)$ be generating function of $\pi \in S_n$ where $\pi(1) = l$ and $\pi(2) < l$.

Then, we have their recurrences,

$$U(l, n, x) = \sum_{i=l}^{n-1} U(i, n-1, x) + D(i, n-1, x),$$

$$D(l, n, x) = \sum_{i=1}^{l-1} U(i, n-1, x) + x \cdot D(i, n-1, x),$$

under the boundary conditions $U(l, n, x) = \begin{cases} 0, & \text{if } n < l; \\ 1, & \text{if } n = 1. \end{cases}$

and $D(l, n, x) = 0$ if $n < l$.

(Continue)

Then, finally, we notice that

$$f_{\varnothing}(n,x) = \sum_{i=1}^{n} U(i,n,x) + D(i,n,x) = U(1,n+1,x).$$

Distribution of this no-restriction case

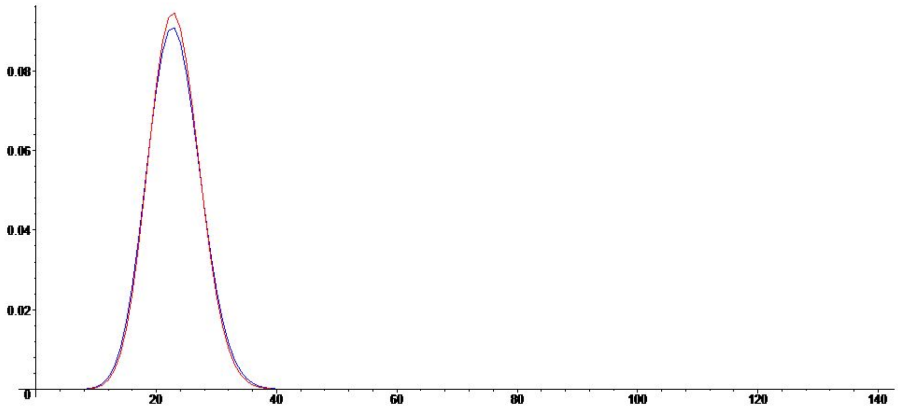


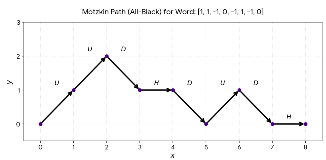
Figure 1: Probability distribution of $f(140)$ and the binomial distribution $Bi(138, \frac{1}{6})$.

3 Generating function of DD on a 321-pattern avoidance permutations

This is the main interest in many papers. It also has a connection with object in algebra, [6].

Before we discuss the generating function, we'd like to give a *"proof from the book"* that the number of 321-avoidance permutation with no DD is M_n, the Motzkin number.

Definition: (Motzkin word, Motzkin number) $w_1 \dots w_n \in \{-1, 0, 1\}^n$ is a *Motzkin word*, if $\sum_{i=1}^{j} w_i$ is ≥ 0 if $j < n$ and 0 if $j = n$. The number of such words is the n-th *Motzkin number*, written M_n.



Outline of the proof

Theorem 1. *(Proof is from [5]): The number of* 321*-avoiding permutations of length* n *without double deficiencies is* M_n*.*

We will give an outline of the proof.

- The 321-avoiding permutations can always be broken down to two increasing sequences. The first sequence comes from "left to right maxima".

  Examples: $\pi_1 = [5, 6, 1, 2, 3, 4, 7, 8], \pi_2 = [3, 4, 1, 5, 2, 7, 6, 8]$.

- Consider V to be the set of values from "left to right maxima" and I to be the set of indices of these values.

  Examples: $I_1 = \{1, 2, 7, 8\}$ and $V_1 = \{5, 6, 7, 8\}$.
  $I_2 = \{1, 2, 4, 6, 8\}$ and $V_2 = \{3, 4, 5, 7, 8\}$

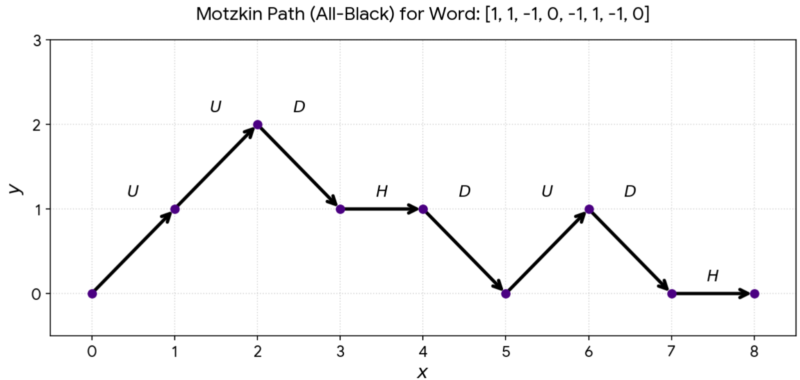
- It can be shown that the set of double deficiency $\mathcal{D}(\pi)$ is the set of numbers (from 1 to n) that are not in $I \cup V$. In these examples, $\mathcal{D}(\pi_1) = \{3, 4\}$ and $\mathcal{D}(\pi_2) = \{\}$.

- Finally, for those permutations that has no DD, we can construct a bijective map to Motzkin word as following:

$$w_i = \begin{cases} 0, & \text{if } i \in I \cap V; \\ 1, & \text{if } i \in I \cap \overline{V}; \\ -1, & \text{if } i \in \overline{I} \cap V. \end{cases}$$

Example: The corresponding Motzkin word of $\pi_2 = [{\color{red}3}, {\color{red}4}, 1, {\color{red}5}, 2, {\color{red}7}, 6, {\color{red}8}]$ with $I_2 = \{1, 2, 4, 6, 8\}$ and $V_2 = \{3, 4, 5, 7, 8\}$ is

$$[1, 1, -1, 0, -1, 1, -1, 0].$$



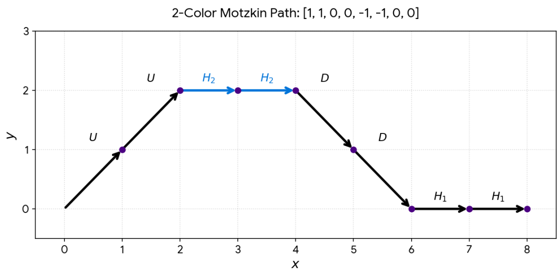
Remark: There are detail that we did not mention, i.e. the inverse map, etc. But once you see the mapping, I hope it is convincing enough that it works.

Now the generating function of DD, $f_{321}(n, x)$

From what we explained, it implies that DD is the blue level (the missing indices) corresponding Motzkin path in the **2-color Motzkin path where the level path on x-axis has no color**. Hence, we will consider the generating function of this Motzkin paths.

Example: The corresponding Motzkin word of $\pi_1 = [5, 6, 1, 2, 3, 4, 7, 8]$ with $I_1 = \{1, 2, 7, 8\}$ and $V_1 = \{5, 6, 7, 8\}$ (the the set of missing indices is $\{3, 4\}$) is

$$[1, 1, 0, 0, -1, -1, 0, 0].$$

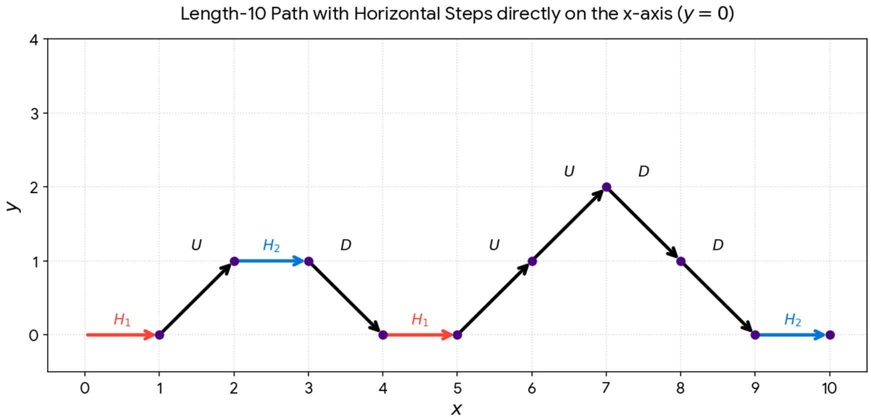


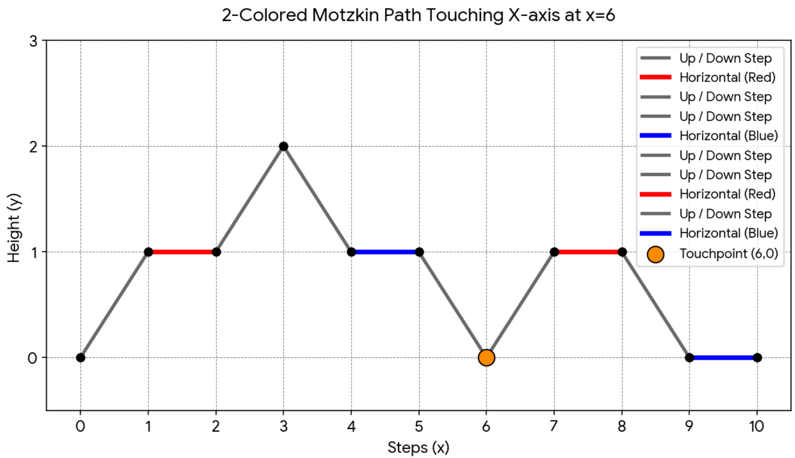
Grand generating function $F_0(z)$

Consider $F_0 := F_0(z)$ be the generating function of Motzkin numbers when the level path has 2-color (including those level paths on the x-axis). We have the relation,

$$F_0 = 1 + (x+1)zF_0 + zF_0zF_0,$$

where x keeps track of number of blue level.





Solving this to get

$$F_0 = \frac{1 - zx - z - \sqrt{1 - 2zx - 2z + z^2x^2 + 2z^2x - 3z^2}}{2z^2}.$$

Grand generating function $F_1(z)$

Generating function of 2-color Motzkin path with level path on x-axis has no color,

$$F_1 = 1 + zF_1 + zF_0zF_1.$$

Then

$$F_1 = \frac{2}{1 + (x-1)z + \sqrt{[1-(x+3)z][1-(x-1)z]}}.$$

The function F_1 is the grand generating function for DD of 321-avoid permutations. That is, we expand F_1 using Taylor series

$$F_1 = f_{321}(0,x) + f_{321}(1,x)z + f_{321}(2,x)z^2 + \dots.$$

This way we can compute the generating function of $n = 17$.

Faster way to compute

The faster way is to compute $f(n) := f_{321}(n, x)$ directly from the recurrences.

Relation of F_0:

$$g(n) = (x+1) \cdot g(n-1) + \sum_{i=2}^{n} g(i-2) \cdot g(n-i),$$

Relation of F_1:

$$f(n) = f(n-1) + \sum_{i=2}^{n} g(i-2) \cdot f(n-i),$$

with initial conditions $f(0) = 1$ and $g(0) = 1$.

This way we can compute the generating function of $n = 150$.

The fastest way to compute

The fastest way is to obtain the **holonomic relation** of f itself. In this case, the relation has order 3 with coefficient of degree 1 in n.

$$\begin{aligned}(n+4)x \cdot f(n+3) = &((5x^2+9x-4)+(2x^2+3x-1)n)f(n+2)\\ &-(x-1)((x^2+8x+5)+(x^2+5x+2)n)f(n+1)\\ &+(x-1)^2(x+3)(n+1)f(n),\end{aligned}$$

with initial conditions $f(0) = f(1) = 1$ and $f(2) = 2$.

With this method, we are able to compute the generating function of $n = 800$.

The distribution of DD for $n = 800$.

The distribution is shown below.

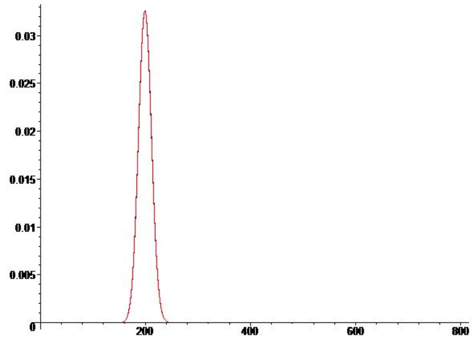


Figure 2: Probability distribution of $f(800)$ and the binomial distribution $Bi(798, \frac{1}{4})$.

The generating function (counting DD) of 321-avoiding permutation converges to binomial distribution with $p = \frac{1}{4}$ and $N = n - 2$.

Proposition 2 (A binomial approximation)**.** *Let X_n denote the number of double deficiencies in a uniformly random permutation in* $\mathrm{Av}_n(321)$*. For* $n \geq 2$*, the support of X_n is* $\{0, 1, \ldots, n-2\}$*, and*

$$\mathbb{E}[X_n] = \frac{(n-1)(n-2)}{2(2n-1)}$$

and

$$\mathrm{Var}(X_n) = \frac{(n+1)(n-2)(3n^2-8n+3)}{2(2n-1)^2(2n-3)}.$$

Moreover,

$$d_{\mathrm{TV}}\left(\mathcal{L}(X_n), \mathrm{Bin}\left(n-2, \frac{1}{4}\right)\right) = O\left(n^{-1/2}(\log n)^{1/4}\right),$$

where the total variation of probability distributions μ and ν on $\mathbb{Z}$ is given by,

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{j \in \mathbb{Z}} |\mu\{j\} - \nu\{j\}|\,.$$

In particular, X_n is asymptotically normal.

Remark: As a corollary, the distribution of the number of blue-color-level in the 2-color Motzkin path (with level path on x-axis has no color) is a binomial distribution.

Last comment

The generating function of DD for 321-avoidance is special because it relates to 2-color Motzkin paths. This is the nicest case!

4 Generating function of DD on permutations that avoid other patterns of length 3

4.1 312-avoidance

Lemma 3. *DD implies the pattern 312.*

Therefore, any 312-avoiding permutation has no DD. Hence, the generating function of DD of 312-avoiding permutation is a constant.

Corollary 4. $f_{312}(n, x) = C(n)$, *i.e.* n^{th} *Catalan number.*

4.2 132-avoidance

For the other 3 pattern avoidances of length 3 (i.e. 132, 231 and 213), we try to set up the recurrences for a fast calculation of generating function of DD.

The usual set up:

It is quite common that, in this situation, we consider the highest number in the n-permutation as a divider for two sides of smaller numbers.

$$L \;\; n \;\; R$$

where all entries in L are larger than all of the entries in R (to avoid 132 pattern). Let's say $\pi(i) = n$;

$$\begin{array}{ccccccccc} 1 & 2 & \ldots & i-1 & i & i+1 & \ldots & n \\ L_1 & L_2 & \ldots & L_{i-1} & n & R_1 & \ldots & R_{n-i}. \end{array}$$

Then the the entries in L is $\{n-i+1, n-i+2, ..., n-1\}$ and the entries in R is $\{1, 2, ..., n-i\}$.

Since all entries in L are larger than any entry in R, there is no DD between the two sets. The only possible occurrence of DD between each part is the entries in L and $\{n\}$.

Reduction and shifting of indexes:

Double deficiency and pattern avoidance are a relative quantity between indexes and values of π. Here we let $f(n,c) := f_{132}(n,c,x)$ be the generating function DD of 132-avoidance of n-permutation but the indexes of permutation is from $c+1$ to $c+n$, i.e. the shift of indexes by c (c is allowed to be negative).

Then L and R themselves can be considered as individual units for generating function.

Example 1: $L = \{n-i+1, n-i+2, ..., n-1\}$ with indexes from 1 to $i-1$. All the entries and index will be shifted by $-(n-i)$. That is $L_{shift} = \{1, 2, \ldots, i-2, i-1\}$ with indexes from $1-(n-i)$ to $i-1-(n-i)$.

That is the generating function corresponds to $f(i-1, -(n-i))$.

Example 2: $R = \{1, 2, ..., n-i\}$ with indexes from $i+1$ to n.

That is the generating function corresponds to $f(n-i, i)$.

Fast recurrences

The recurrence of $f(n,0) := f_{132}(n,0,x)$ goes as

$$f(n,c) = \sum_{i=1}^{n} x^{\Delta_i} f(i-1, c-(n-i)) \cdot f(n-i, c+i),$$

where $f(0,c) = 1$ and $\Delta_i = 1$ if $n \in \{c+1, \ldots, c+i-1\}$ and 0, otherwise.

Distribution of DD for $n = 140$.

With this simple recurrences, we calculate the probability generating function for $n = 140$ shown below.

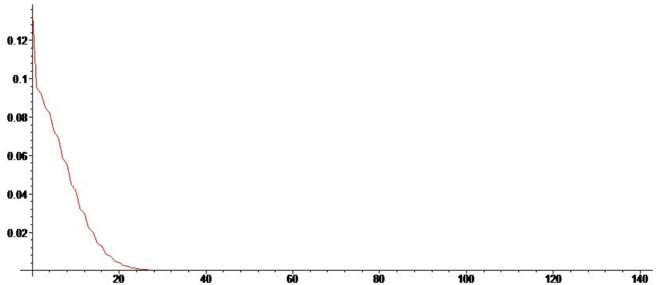


Figure 3: Probability distribution of $f(140, 0)$.

The plot does not look as nice as other case of pattern avoidance, so we also plot the average number of DD for each n.

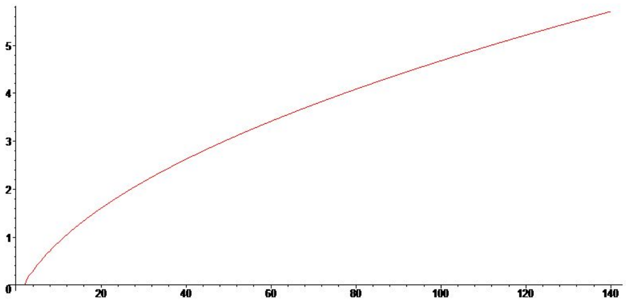


Figure 4: Mean numbers of DD for 132-avoidance in n-permutation, $0 \le n \le 140$.

The generating function of DD for 132-avoiding permutations is special because it has the strangest distribution.

4.3 213-avoidance

In the paper, we showed that 213-avoidance and 132-avoidance are in the same *DD-Wilf equivalent*. That is the generating functions of DD of both cases are identical.

4.4 231-avoidance

We use the same set up to 132-avoidance. The largest number in the n-permutation is used a divider for two sides of smaller numbers.

$$L \;\; n \;\; R$$

where, this time, all entries in L is smaller than all of the entries in R (to avoid 231 pattern). Let's say $\pi(i) = n$. Then, the entries in L are $\{1, 2, \ldots, i-1\}$ and the entries in R are $\{i, i+1, \ldots, n-1\}$.

Fast recurrences

We let Δ to keep track of DD between L and $\{n\} \cup R$. We found that

$$\Delta = \min(c, i-1, n-c, n-i+1).$$

The recurrence of $f(n,c) := f_{231}(n,c,x)$ goes as

$$f(n,c) = \sum_{i=1}^{n} x^{\Delta} f(i-1,c) \cdot f(n-i,c+1),$$

where $f(n,c) = C_n$ (i.e. the n^{th} Catalan number), for $n \leq c$.

Distribution of DD for $n = 120$.

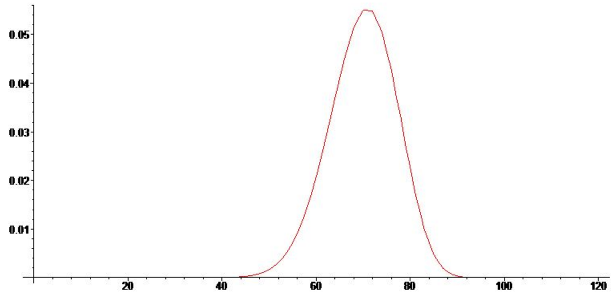


Figure 5: Probability distribution of $f(120, 0)$.

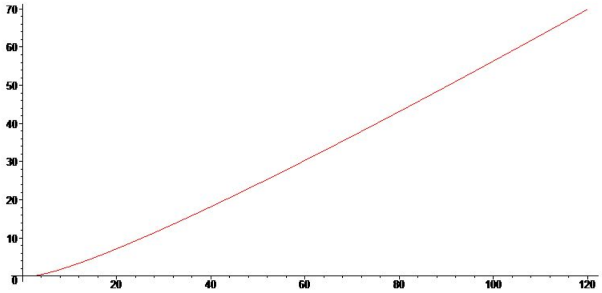


Figure 6: Mean numbers of DD for 231-avoidance in n-permutation, $0 \leq n \leq 120$.

The generating function of DD for 231-avoidance is special because it has the highest numbers of DD among all 6 pattern avoidance of length 3.

4.5 123-avoidance

This is the only case in pattern of length 3 that we could not do.

Open problem 1:

Find a fast way to compute the generating function of DD of 123-avoidance case.

(Prize: 100$ from DZ, 50$ from TT and 50$ from TK donate to OEIS).

Discussion of 123-avoidance

We will try to make some discussion despite the fact that we could not solve it.

Let $f(n, c) := f_{123}(n, c, x)$, i.e. the generating function of DD for n-permutation that avoids pattern 123.

Then we can find the recurrence of $f(n, c)$ as

$$f(n, c) = \sum_{i=1}^{n} g(i, c - (n - i)) \cdot f(n - i, i + c),$$

where $g(n, c)$ is the generating function of DD over $P(n)$.

Remark: this recurrence is an analog of the recurrence for 2-color Motzkin path that we used for case 321-avoidance.

$P(n)$ is the prefix restriction of n-permutation with 123-avoidance i.e. a complete block of the highest possible numbers.

Examples

$$
\begin{aligned}
&P(1) = \{[1]\},\\
&P(2) = \{[1,2]\},\\
&P(3) = \{[1,3,2],[2,1,3]\},\\
&P(4) = \{[1,4,3,2],[2,1,4,3],[2,4,1,3],[3,1,4,2],[3,2,1,4]\}.
\end{aligned}
$$

We note that $|P(n)| = C(n-1)$, i.e. $(n-1)^{th}$ Catalan number.

I also spent time looking at B_1= set of "Left to Right Minima" and B_2= set of "Right to Left maxima" on $P(n)$.

However, with all that said, we did not manage to find a recurrence of $g(n,c)$.

5 DD with 2 Patterns-avoidance

We also investigate the generating of DD from permutation that avoids-2-patterns of length 3.

We manage to solve all of the cases

$$\{321, 231\}, \quad \{321, 132\}, \quad \{132, 231\}, \quad \{132, 123\}, \quad \{132, 213\}, \quad \{123, 231\}.$$

(The other cases are either equivalent to one of these cases or not interesting. For example, $f_{\{321,123\}}(n, x) = 0$ for $n \geq 5$.)

For each case, the generating function of DD satisfies C-finite relation.

We will give only one example here.

Avoiding S where $S = \{[3,2,1],[2,3,1]\}$.

Since S contains pattern $[2,3,1]$, let's consider $\pi(i) = n$ as a divider. The permutation $\pi \in S$ looks like

$$\begin{array}{cccccccccc} 1 & 2 & \dots & i-1 & i & i+1 & i+2 & \dots & n \\ L_1 & L_2 & \dots & L_{i-1} & n & i & i+1 & \dots & n-1, \end{array}$$

where $L = \{1, 2, \dots, i-1\}$.

All the values in the right part (except the last entry) give new DD. There are $n - i - 1$ of them.

Consider $f_S(n,x) =: f_S(n)$. The recurrence for f where the initial condition $f_S(0) = 1$ is

$$f_S(n) = f_S(n-1) + \sum_{i=1}^{n-1} x^{n-i-1} f_S(i-1).$$

We note that $f_S(n,1) = 2^{n-1}$.

C-finite relation: $f_S(n+2) = (x+1)f_S(n+1) - (x-1)f_S(n)$.

Distribution of DD for $n = 170$.

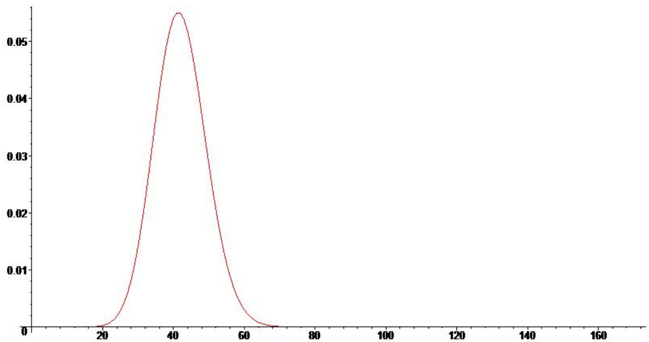


Figure 7: Probability distribution of $f_S(170)$.

It can be shown from the recurrence that the mean number of DD for permutation of length n in this class is $\dfrac{n-2}{4}$.

Last words

Several questions remain. A comparable recurrence for the double-deficiency generating functions of 123-avoiding permutations has not been obtained (Open Problem 1). The asymptotic behavior of the distributions for the remaining single-pattern classes is also open: the data suggest that the distribution for Av(231) is nearly symmetric, whereas the one for Av(132) is strongly skewed, with a mean that grows sublinearly in n rather than linearly as in Av(321) and Av(231). Finally, it would be natural to investigate double deficiencies in avoidance classes defined by patterns of length 4 (especially 4321) and, more generally, longer patterns.

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