von Neumann Poker with Finite Decks

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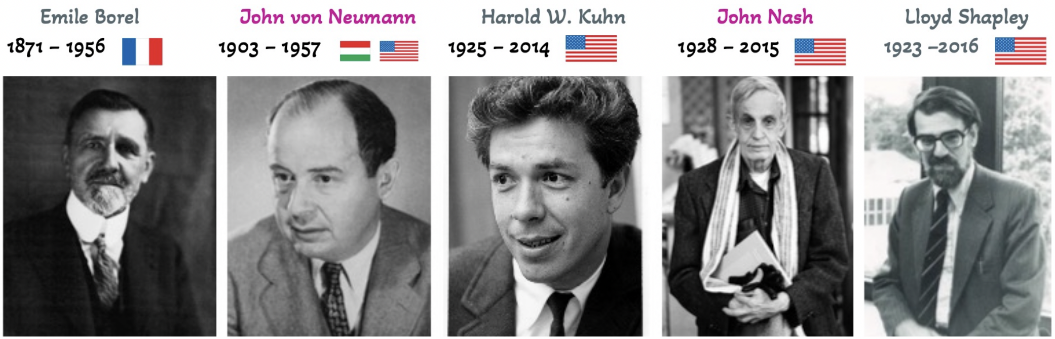
REU Mathematics Seminar

Poker x Math

- Pioneer mathematicians in *Game Theory/Poker* include Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley.
- They believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking.
- von Neumann and Morgenstern wrote the bible of game theory “Theory of Games and Economic Behavior”, first published in 1944.



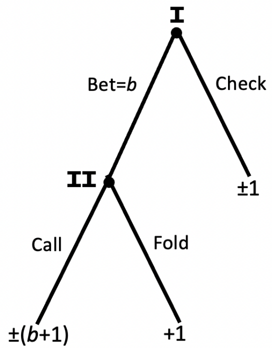
Outline

- Summary of von Neumann Poker (1938): 2-player, continuous
- Game theory crash course
- von Neumann Poker: 2-player, discrete

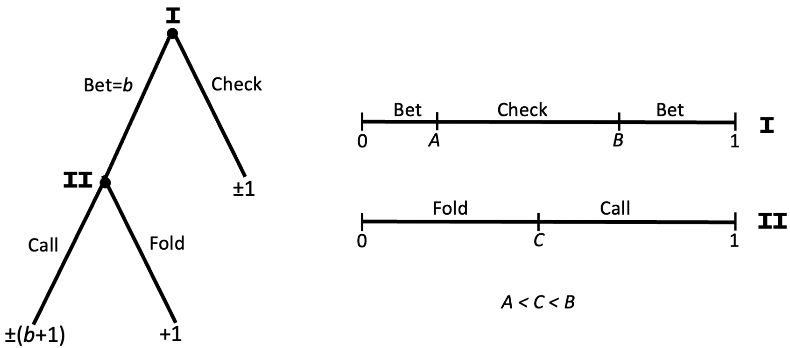
1. von Neumann Poker

- In 1938, John von Neumann proposed his now-famous mathematical model of poker, a game with an *uncountably infinite* deck.
- Player I and Player II are dealt (uniformly at random) two "cards", **real numbers** x**,** $y \in [0, 1]$.
- They each see their own card, but have no clue about the opponent's card.
- At the start they each **put $1 into the pot** (the so called *ante*).
- Player I has the option to check or bet $\$b$, while Player II can only call or fold.

von Neumann's pure Nash Equilibrium



von Neumann proved that the following pair of strategies is a pure *Nash Equilibrium (NE)*, i.e. if the players both follow their chosen strategy, neither of them can do better (on average) by doing a different strategy.



When $b = 2$, the advice is as follows:

- **Player I:** if $0 < x < \frac{1}{9}$ or $\frac{7}{9} < x < 1$ you should **bet**, otherwise **check**.
- **Player II:** If $0 < y < \frac{5}{9}$ you should **fold**, otherwise **call**.
- The expected value, i.e. the value of the game (for Player I) is $\frac{1}{9}$.

2. Game Theory Crash Course

Example 1: Friend or Foe

Each player attempts to persuade the other to trust them, after which they secretly vote “friend” or “foe”.

- If both vote “friend”, they split the trust fund evenly.
- If one votes “friend” and the other “foe”, the foe collects the entire trust fund and the friend receives nothing.
- If both vote “foe”, neither player wins any money.

The pay-off matrix of Friend or Foe:

	B friend	B foe
A friend	(500K, 500K)	(0,1M)
A foe	(1M,0)	(0,0)

What would you play?

Zero-sum Game

The main interest of this talk is the *zero-sum game*. The poker game that we introduced is a zero-sum game.

The **zero-sum game** is the game where the entries in each cell add up to 0. Collaboration does not give any advantageous in a zero-sum game (while it does in the non-zero sum game).

Example 2

	Player II plays 1	Player II plays 2	Player II plays 3
Player I plays 1	(3, -3)	(-4, 4)	(2, -2)
Player I plays 2	(-1, 1)	(4, -4)	(-2, 2)
Player I plays 3	(-3, 3)	(1, -1)	(4, -4)
Player I plays 4	(1, -1)	(-1, 1)	(1, -1)

A pay-off matrix of the zero-sum game is written from Player 1's point of view only:

$$\begin{vmatrix} 3 & -4 & 2 \\ -1 & 4 & -2 \\ -3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

Important! The pay-off of Player II in each entry is the negative of the entry.

Play-safe strategies for the zero-sum game

- For Player I (row): the row maximin.
- For Player II (column): the column minimax. *(Player II aims to minimize their expected loss, or equivalently the expected gain of Player I.)*

$$\begin{vmatrix} 3 & -4 & 2 \\ -1 & 4 & -2 \\ -3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

What are the play-safe strategies for Player I and II?

Pure Nash Equilibrium is a situation in a game where no player can benefit by changing their pure strategy while the other players keep theirs unchanged.

This concept is important because this strategy pair can be considered **stable** as neither player has an incentive to deviate from his choice.

Theorem 1. *In a zero-sum game there will be a pure NE if and only if*

$$\textit{the row maximin} = \textit{the column minimax.}$$

Do we have a pure NE in this example?

Example 3: There is a pure NE

The following pay-off matrix has a pure NE

$$\begin{vmatrix} 4 & -1 & 2 & 3 \\ 4 & 6 & 3 & 7 \\ 1 & 2 & -2 & 4 \end{vmatrix}$$

A mixed strategy for a game with no stable solution

When the game does not have a stable solution, it makes sense to not stick to a particular choice, as your opponent could take an advantage of it. Here the idea of mixed strategy (play each choice randomly with particular probability) comes into play.

Example 4: Matching pennies

A and B plays a game called matching pennies. A and B choose the side of the coin separately. If they are the same A gains 1 point and B loses 1 point. If they are the different B gains 1 point and A loses 1 point.

What is your strategy to play this game?

	H	T
H	1	-1
T	-1	1

Example 5: Another Matching pennies

We still a consider a matching pennies. However the pay-off matrix has been modified.

	H	T
H	100	-1
T	-1	1

What is your recommended strategy to play this game now?

➽ Mixed NEs

Mixed Strategy: A strategy where a player randomizes over two or more pure strategies, assigning a probability to each option.

von Neumann's Theorem (1928): Every finite two-person zero-sum game has at least one Nash equilibrium in mixed strategies. They are the maximin mixed strategies.

Linear Programming for Mixed NE

Let p_1 and p_2 be the probabilities that Player A plays H and T accordingly.

Maximize v_1
subject to the constraints:

$$\begin{aligned}
\text{Player B plays H:} \quad 100p_1 + (-1)p_2 &\geq v_1, \\
\text{Player B plays T:} \quad (-1)p_1 + (1)p_2 &\geq v_1, \\
p_1 + p_2 &= 1, \\
p_1, p_2 &\geq 0.
\end{aligned}$$

That is we maximize the minimum expected payoff for each choice of Player B.

The solution is $p_1 = \dfrac{2}{103}$, $p_2 = \dfrac{101}{103}$ and $v_1 = \dfrac{99}{103}$.

3. von Neumann Poker, 2-player discrete

Question: How can we construct a payoff matrix with n cards?

- A strategy for Player I can be *any* subset, S_1, of $\{1, \ldots, n\}$, that advises: 'If your card belongs to S_1 you should **bet**, otherwise, **check**'.
- Similarly a strategy for Player II, S_2, can be any such subset, that tells her to '**call** if her card $j \in S_2$, otherwise **fold**'.
- Thus, the payoff matrix can be obtained by listing outcomes of all pairs $[S_1, S_2]$
- Once constructed, we look for pure NEs in the usual way:

"If the row maximin equals the column minimax, then NEs exist."

Example: Payoff Matrix for $n = 2$ cards, Bet size $b = 2$

Player II is the Column Player. She can either Call of Fold.

Paytable for 2 cards: {1,2}, and with bet size b=2.

Player I is the Row Player.

He can either Bet or Check.

Strategy	$S_2 = \{\ \}$ Always Fold	$S_2 = \{1\}$ Call if "1", Fold if "2"	$S_2 = \{2\}$ Call if "2", Fold if "1"	$S_2 = \{1,2\}$ Always Call	Row Min
$S_1 = \{\ \}$ Always Check	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	
$S_1 = \{1\}$ Bet if "1", Check if "2"	(+1+1)/2 = 1	(+1+1)/2 = 1	(-3+1)/2 = -1	(-3+1)/2 = -1	
$S_1 = \{2\}$ Bet if "2", Check if "1"	(-1+1)/2 = 0	(-1+3)/2 = 1	(-1+1)/2 = 0	(-1+3)/2 = 1	
$S_1 = \{1,2\}$ Always Bet	1	(+1+3)/2 = 2	(-3+1)/2 = -1	(-3+3)/2 = 0	
Column Max					

Example: Payoff Matrix for $n = 2$ cards, Bet size $b = 2$

Player II is the Column Player. She can either Call of Fold.

Player I is the Row Player.

He can either Bet or Check.

Strategy	$S_2 = \{\ \}$ Always Fold	$S_2 = \{1\}$ Call if "1", Fold if "2"	$S_2 = \{2\}$ Call if "2", Fold if "1"	$S_2 = \{1,2\}$ Always Call	Row Min
$S_1 = \{\ \}$ Always Check	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	(-1+1)/2 = 0	0
$S_1 = \{1\}$ Bet if "1", Check if "2"	(+1+1)/2 = 1	(+1+1)/2 = 1	(-3+1)/2 = -1	(-3+1)/2 = -1	-1
$S_1 = \{2\}$ Bet if "2", Check if "1"	(-1+1)/2 = 0	(-1+3)/2 = 1	(-1+1)/2 = 0	(-1+3)/2 = 1	0
$S_1 = \{1,2\}$ Always Bet	1	(+1+3)/2 = 2	(-3+1)/2 = -1	(-3+3)/2 = 0	-1
Column Max	1	2	0	1	

Paytable for 2 cards: {1,2}, and with bet size b=2.

Row maximin = Column minimax = 0

So there are TWO pure NEs.

In both of them,
- Player II calls if her card is "2" and folds if her card is "1", while

- Player I always checks in the first strategy, and checks if his card is "1" in the second strategy.

This is not very interesting, since the expected gain (value of the the game) is 0.

Payoff Matrix for $n = 4$ cards, Best size $b = 2$

Strategy	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Row Min
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	1/2	1/2	1/6	1/6	1/6	1/6	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/2	-1/2	
3	1/3	1/2	1/3	0	0	1/2	1/6	1/6	0	0	-1/3	1/6	1/6	-1/6	-1/3	-1/6	
4	1/6	1/3	1/3	1/6	-1/6	1/2	1/3	0	1/3	0	-1/6	1/2	1/6	0	0	1/6	
5	0	1/6	1/6	1/6	0	1/3	1/3	1/6	1/3	1/6	1/6	1/2	1/3	1/3	1/3	1/2	
6	5/6	1	1/2	1/6	1/6	2/3	1/3	1/3	-1/6	-1/6	-1/2	0	0	-1/3	-5/6	-2/3	
7	2/3	5/6	1/2	1/3	0	2/3	1/2	1/6	1/6	-1/6	-1/3	1/3	0	-1/6	-1/2	-1/3	
8	1/2	2/3	1/3	1/3	1/6	1/2	1/2	1/3	1/6	0	0	1/3	1/6	1/6	-1/6	0	
9	1/2	5/6	2/3	1/6	-1/6	1	1/2	1/6	1/3	0	-1/2	2/3	1/3	-1/6	-1/3	0	
10	1/3	2/3	1/2	1/6	0	5/6	1/2	1/3	1/3	1/6	-1/6	2/3	1/2	1/6	0	1/3	
11	1/6	1/2	1/2	1/3	-1/6	5/6	2/3	1/6	2/3	1/6	0	1	1/2	1/3	1/3	2/3	
12	1	4/3	5/6	1/3	0	7/6	2/3	1/3	1/6	-1/6	-2/3	1/2	1/6	-1/3	-5/6	-1/2	
13	5/6	7/6	2/3	1/3	1/6	1	2/3	1/2	1/6	0	-1/3	1/2	1/3	0	-1/2	-1/6	
14	2/3	1	2/3	1/2	0	1	5/6	1/3	1/2	0	-1/6	5/6	1/3	1/6	-1/6	1/6	
15	1/2	1	5/6	1/3	-1/6	4/3	5/6	1/3	2/3	1/6	-1/3	7/6	2/3	1/6	0	1/2	
16	1	3/2	1	1/2	0	3/2	1	1/2	1/2	0	-1/2	1	1/2	0	-1/2	0	
Column Max																	

Paytable for 4 cards: {1,2,3,4}, and with bet size b=2.

Strategy	Player I bets if... / Player II calls if...
1	{}
2	{1}
3	{2}
4	{3}
5	{4}
6	{1, 2}
7	{1, 3}
8	{1, 4}
9	{2, 3}
10	{2, 4}
11	{3, 4}
12	{1, 2, 3}
13	{1, 2, 4}
14	{1, 3, 4}
15	{2, 3, 4}
16	{1, 2, 3, 4}

For the bet size $b = 2$, and consider the pure NEs for other n cards.

- If the card has only 2 cards, `vnNE(2,2);` gives

$$[\phi, \{2\}] \text{ and } [\{2\}, \{2\}]$$

- `vnNE(3,2);` is equally boring, giving the two trivial pairs $[\phi, \{3\}]$ and $[\{3\}, \{3\}]$
- `vnNE(4,2);`, `vnNE(5,2;)`, and `vnNE(6,2);` are even more boring, they are empty! That is, there is no pure NEs.

```
> vnNE(2,2);
                    {[{ }, {2}, 0], [{2}, {2}, 0]}
> vnNE(3,2);
                    {[{ }, {3}, 0], [{3}, {3}, 0]}
> vnNE(4,2);
                                { }
> vnNE(5,2);
                                { }
> vnNE(6,2);
                                { }
```

We will show the solution of mixed strategy for $n = 4, 5, 6$ through Maple program.

But now comes a nice surprise, `vnNE(7,2);` gives three pure, *non-trivial*, NEs.

- For all of them Player I bets if his card belongs to $\{1, 6, 7\}$. Player II calls if her card is in $\{3, 6, 7\}$, $\{4, 6, 7\}$, or $\{5, 6, 7\}$. The value of the game is $\frac{2}{21}$.
- So with 7 cards we already have bluffing! If Player I has the card labeled 1, he should bet even though he would definitely lose the bet if Player II calls.

```
> vnNE(7,2);
```

$$\left\{\left[\{1,6,7\},\{3,6,7\},\frac{2}{21}\right],\left[\{1,6,7\},\{4,6,7\},\frac{2}{21}\right],\left[\{1,6,7\},\{5,6,7\},\frac{2}{21}\right]\right\}$$

Moving right along, `vnNE(8,2);` also gives you three pure NEs.

- For all of them Player I bets if his card belongs to $\{1, 7, 8\}$, but Player II calls if her card is in either $\{4, 7, 8\}$, $\{5, 7, 8\}$, or $\{6, 7, 8\}$. The value of the game is $\frac{3}{28}$, getting tantalizingly close to von Neumann's $\frac{1}{9}$.

```
> vnNE(8,2);
```

$$\left\{\left[\{1,7,8\},\{4,7,8\},\frac{3}{28}\right],\left[\{1,7,8\},\{5,7,8\},\frac{3}{28}\right],\left[\{1,7,8\},\{6,7,8\},\frac{3}{28}\right]\right\}$$

➽ Curse of dimensionality!

Since the size 2^n by 2^n of the payoff matrix grows exponentially, and we did not make *any plausibility assumptions*, there is only so far we can go with this naive ***vanilla*** approach.

But nine cards, with 512×512, paytable are still doable.

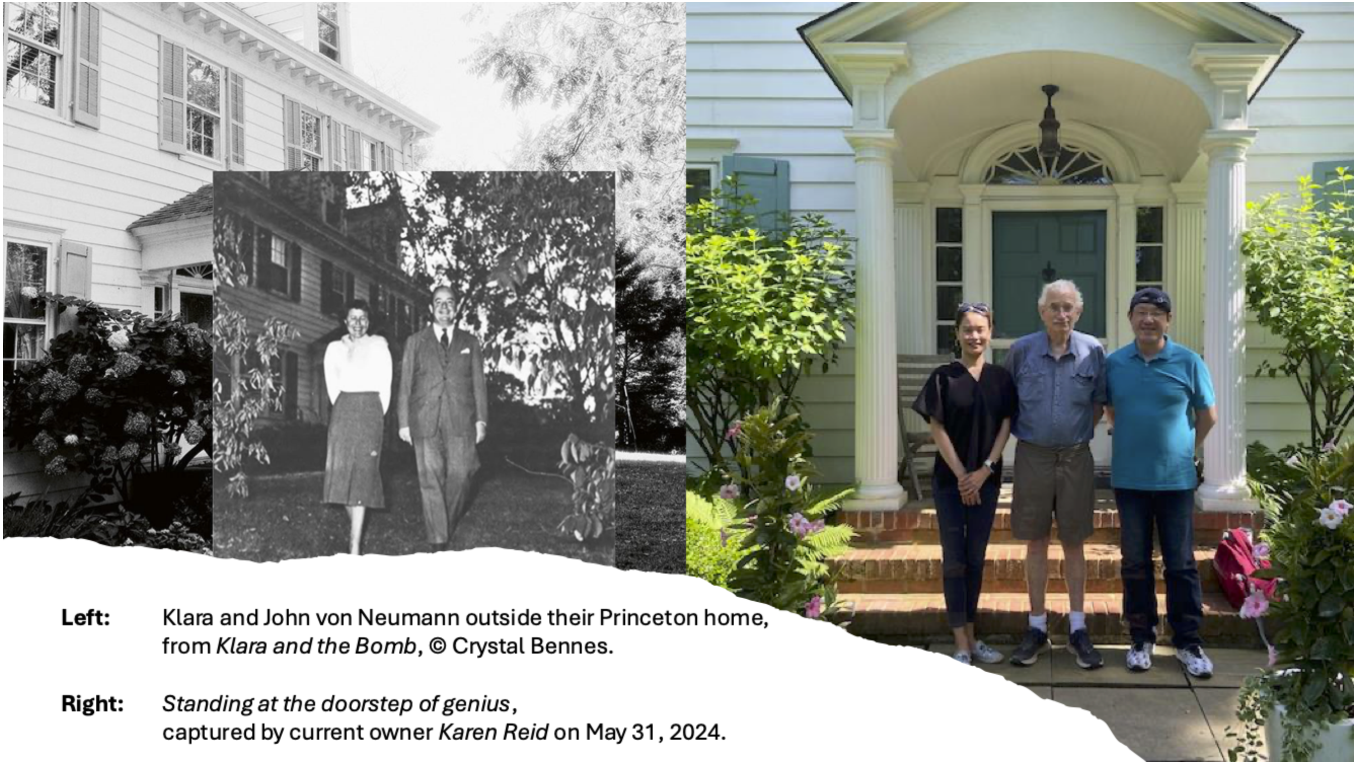
Indeed, `vnNE(9,2);` gives you seven pure NEs in this case.

- For all of them $S_1 = \{1, 8, 9\}$, but Player II has seven choices, all with four members, including, of course, $\{6, 7, 8, 9\}$.

```
> vnNE(9,2);
```

$$\{[\{1,8,9\},\{3,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{3,7,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{4,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{4,7,8,9\},\tfrac{1}{9}],$$
$$[\{1,8,9\},\{5,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{5,7,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{6,7,8,9\},\tfrac{1}{9}]\}$$

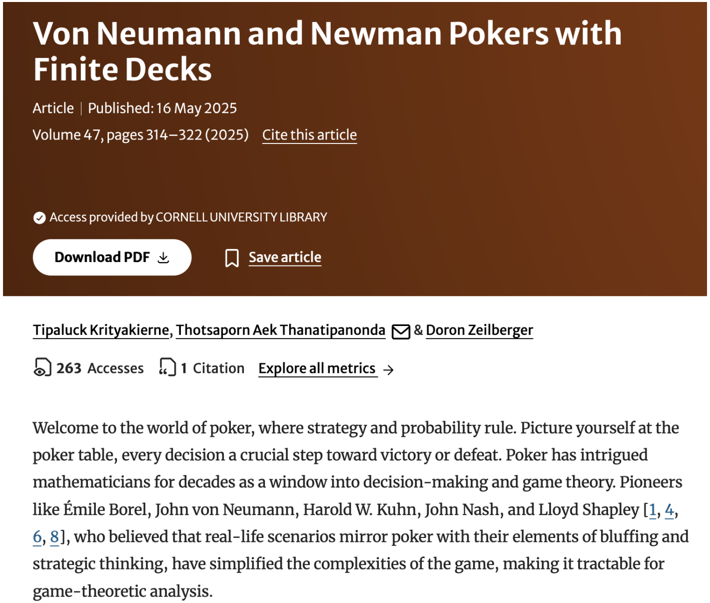
Then & Now: von Neumann's House



Left: Klara and John von Neumann outside their Princeton home, from *Klara and the Bomb*, © Crystal Bennes.

Right: *Standing at the doorstep of genius*, captured by current owner *Karen Reid* on May 31, 2024.

If you are interested, this work has been published in *The Mathematical Intelligencer*. In addition to the classical two-player von Neumann poker, the paper also studies Newman's poker, where the first player may choose the bet size, and extends the analysis to three-player poker games.



Von Neumann and Newman Pokers with Finite Decks

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Welcome to the world of poker, where strategy and probability rule. Picture yourself at the poker table, every decision a crucial step toward victory or defeat. Poker has intrigued mathematicians for decades as a window into decision-making and game theory. Pioneers like Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley [1, 4, 6, 8], who believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking, have simplified the complexities of the game, making it tractable for game-theoretic analysis.