The Relation Between Card Shuffling and Random Card Color

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Abstract

Over the past few decades, the Card Guessing Game has been the subject of extensive study, with attention given to optimal strategies and the distribution of correct guesses. We introduce a fundamental feature of the game, the Gilbert–Shannon–Reeds (GSR) model of riffle shuffle. For the guessing part, we discuss both the feedback version (the guesses are adjusted according to the cards that they see) and the non-feedback version (the player commits to a complete sequence of guesses in advance).

For the Card Guessing Game with feedback version of $k, k \geq 2$, shuffles, it seems that we can apply the random card coloring to approximate the average number of correct guesses under the optimal guessing strategy. So we will spend time discussing that.

Part I

The Gilbert–Shannon–Reeds (GSR) model of Riffle Shuffle

The process of riffle shuffle goes as followings:

1. (**Starting Deck**) The original deck composes of card from 1 to n, where all the numbers are sorted with number 1 is on top and number n is on the bottom.

2. (**Process for each Riffle Shuffle**) Split the deck into two piles. The probability of cutting the top t cards is $\frac{\binom{n}{t}}{2^n}$.

3. (**Process for each Riffle Shuffle**) Then, interleave the piles back into a single one. Each interleaving has probability $\frac{1}{\binom{n}{t}}$ to happen given that the top part has t cards.

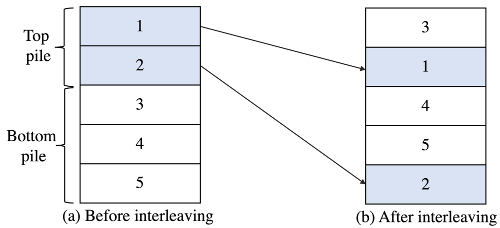
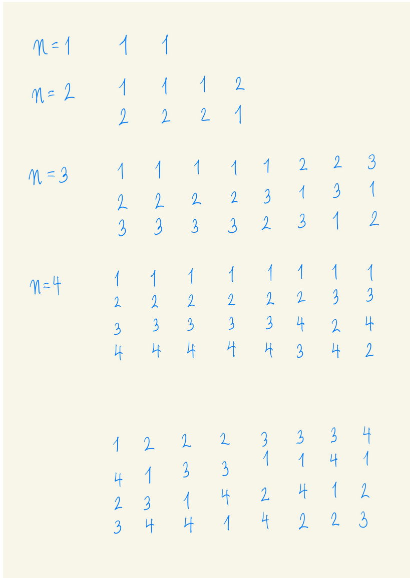


Figure 1: Example of 1-time riffle shuffle of a deck of 5 cards

4. We then apply this process k times for k shuffles.

Examples: 1-time riffle shuffle with n cards



Total number of outcomes for 1-time riffle shuffle is 2^n. Each of the outcome has $\frac{1}{2^n}$ chance to happen.

Part II

Famous Result on Riffle Shuffle by Persi Diaconis

Question: "How many times should a deck of cards be shuffled to mix it up?"

The answer depends on what type of shuffle we are talking about. But for riffle shuffle, Persi Diaconis has the famous result[2, 3, 5] that 7 riffle shuffles are sufficient to get a deck of $n = 52$ cards very close to random, while 6 riffle shuffles do not suffice.

Probabilists have cooked up the *variation distance* as a measure of randomness.

The *variation distance* between two probability distributions Q_1 and Q_2 is defined as

$$||Q_1 - Q_2|| = \frac{1}{2} \sum_{\pi} |Q_1(\pi) - Q_2(\pi)|.$$

Two examples are our starting distribution E, which is given by

$$E(id) = 1,$$
$$E(\pi) = 0, \text{ otherwise,}$$

and the uniform distribution U given by

$$U(\pi) = \frac{1}{n!} \text{ for all permutation } \pi.$$

For our consideration, “being close to random” will be interpreted as “having small variation distance from the uniform distribution.”

The variation distance between E and U is very close to 1. Not random!

$$||E - U|| = 1 - \frac{1}{n!}.$$

The question is which k does $d(k) := ||Rif^{*k} - U||$ is small enough?

In [3], Bayer and Diaconis found a way to calculate $d(k)$ and claimed that $d(7)$ is enough for practical purpose for the shuffled deck of cards to be random.

k	1	2	3	4	5	6	7	8	9	10
$d(k)$	1.000	1.000	1.000	1.000	0.952	0.614	0.334	0.167	0.085	0.043

Part III

Example of Card Guessing with Optimal Strategy

Simple Example for Card Guessing

- Not a card shuffling. The starting deck has red and blue cards with n cards each. These cards are mixed randomly in the deck.
- *Guessing with feedback.* You guess the color of each card one at a time. For each card that you guessed, the moderator will show you the color of that card, so you know whether your guess is correct or not. Then you use that information to help making a decision of the guess for the next card.

Questions

- What is the optimal strategy?
- What is the expected number of cards that you guess the color correctly?

Solutions

The optimal strategy is clear. For each card, you will guess the color that have more cards left in the deck. In case there are equal number of cards in each color left in the deck, you can randomly choose any of the color.

The average number of correct guesses, [11], is

$$S(n) = n + \frac{1}{2}\left(\frac{4^n}{\binom{2n}{n}} - 1\right) = n + \frac{1}{2}\left(\sqrt{\pi n} - 1\right) + O\left(\frac{1}{\sqrt{n}}\right).$$

That is $S(26) = 30.04066478$ and $S(100) = 108.3733540$.

Part IV

Card Guessing after Riffle Shuffle, No-feedback Version

Objective:

Maximize the number of correct guesses of the card values.

Rules:

- The game begins with a deck of n cards, initially ordered as $1, 2, 3, \ldots, n$.
- The dealer performs k riffle shuffles.
- The player then makes one guess for each position in the deck.
- No feedback is provided during or after the guessing phase.
- Make the guess of all n positions all at once.

As shuffle is a random process, for each position i, we would like to guess the number of card that is most likely to show up in that position.

Probability Matrix for 1-shuffle

$a(i, j, n)$ represents the probability that card j appears in position i after a single riffle shuffle. This quantity admits the following closed-form expression:

$$a(i, j, n) = \frac{1}{2^i}\binom{i-1}{j-1} + \frac{1}{2^{n-i+1}}\binom{n-i}{j-i}. \tag{1}$$

For instance, when $n = 5$, the matrix P is given by

$$P = \frac{1}{2^5}\begin{bmatrix} \boxed{\mathbf{17}} & 4 & 6 & 4 & 1 \\ 8 & \boxed{\mathbf{10}} & 6 & 6 & 2 \\ 4 & 8 & \boxed{\mathbf{8}} & 8 & 4 \\ 2 & 6 & 6 & \boxed{\mathbf{10}} & 8 \\ 1 & 4 & 6 & 4 & \boxed{\mathbf{17}} \end{bmatrix}.$$

Probability Matrix for k-shuffle

As P is a transition matrix, P^k gives the probability that card j appears in position i after a k-riffle shuffle. For example for $n = 5$ and 2 shuffle, P^2 is given by

$$P^2 = \frac{1}{(2^5)^2}\begin{bmatrix} \boxed{\mathbf{354}} & 184 & 204 & 184 & 98 \\ \boxed{\mathbf{254}} & 224 & 204 & 208 & 134 \\ 184 & \boxed{\mathbf{224}} & 208 & \boxed{\mathbf{224}} & 184 \\ 134 & 208 & 204 & 224 & \boxed{\mathbf{254}} \\ 98 & 184 & 204 & 184 & \boxed{\mathbf{354}} \end{bmatrix}.$$

That is the strategy for card guessing with no feedback with k-riffle shuffle can be read off from the matrix P^k directly.

The main result of the guessing with feedback version is by Ciucu [4] in 1998.

Theorem (Ciucu [4]). For $k \geq 2\log_2(2n) + 1$, the optimal no-feedback guessing strategy after k riffle shuffles of a deck of $2n$ cards is to guess 1 at the first n positions and $2n$ at the remaining n positions.

Part V

Card Guessing after 1-riffle Shuffle, Feedback Version

Original deck is $[1, 2, 3, \ldots, n]$.

Rule: Riffle shuffle k times, guess card $\rightarrow$ reveal card, then repeat.

Goal: Try to make the correct guesses as many as possible.

We discuss the optimal guessing strategy and statistics of the number of correct guesses.

Note: After 1-riffle shuffle, the deck of card is composed of at most 2 increasing subsequences.

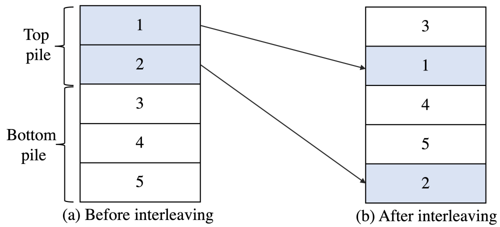


Figure 2: Example of 1-time riffle shuffle of a deck of 5 cards

Algorithm:

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1:Start by guessing number 1.
2:If true then continue to guess the next number in line.
3:If false then the deck is now split into two increasing subsequences.
Guess the first element in the longer subsequence.
4:Continue to guess this way until until no cards remain.
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This algorithm is proved to provide the maximum expected number of correct guesses.

Example:

Permutation π	1	1	1	1	1	2	2	2	3	3	3	4
	2	2	3	3	4	1	3	3	1	4	1	1
	3	4	2	4	2	3	1	4	2	1	4	2
	4	3	4	2	3	4	4	1	4	2	2	3
$P(\pi)$	5/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
#Correct guesses	4	3	3	2	3	2	3	2	3	2	2	3

Figure 3: All possible permutations after shuffling a 4-card deck once. The color indicates a correct guess under the optimal strategy.

The generating function of the number of correct guesses

$$D_n(q) = \sum_{i=0}^{\infty} a_i q^i,$$

where a_i denotes the number of permutations with i correct guesses.

Recurrence:

$$D_n(q) = \underbrace{qD_{n-1}(q) + q^n}_{\text{the first card } = 1} + \underbrace{\sum_{i=0}^{n-2} F(n-1-i, i; q)}_{\text{the first card } > 1}, \qquad \text{(Main recurrence)}$$

where $D_0(q) = 1$.

Remark 1: $F(m, n; q)$ is also a generating function keeping track of number of correct guesses when the length of the two increasing subsequences are known.

Example

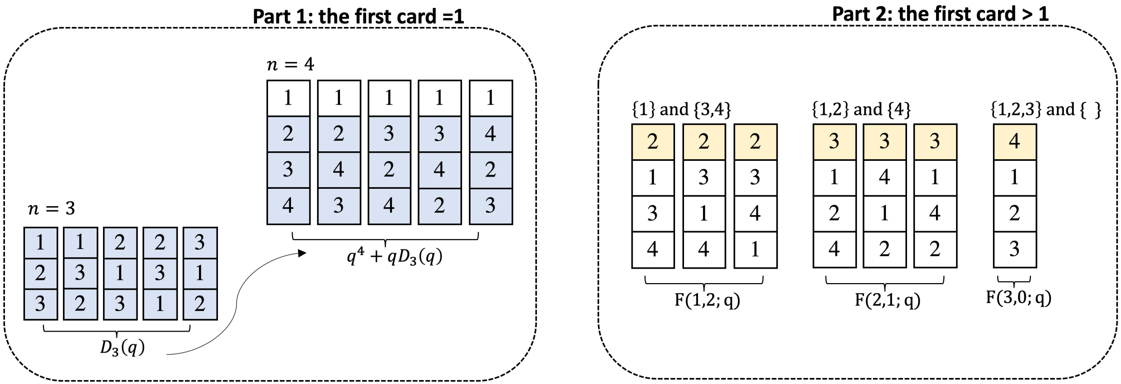

Figure 4: Recurrence structure $D_4(q) = \left(q^4 + qD_3(q)\right) + F(1, 2; q) + F(2, 1; q) + F(3, 0; q)$

Remark 2: The generating function has been computed in 2 parts: Part 1 when the numbers of the cards in each increasing subsequence are not known yet, and Part 2 when the numbers of the cards in each increasing subsequence are already known.

For Part 2, $F(m, n; q)$ can be computed by some recurrence as well.

$$F(m, n; q) = \underbrace{qF(m-1, n; q)}_{\text{next card from longer subsq.}} + \underbrace{F(m, n-1; q)}_{\text{next card from shorter subsq.}}, \quad (2)$$

for $m \geq n$, where $F(m, 0; q) = q^m$. Also, $F(m, n; q) := F(n, m; q)$ whenever $m < n$.

The distribution of number of corrected guesses of n cards

These calculations are from the computation of $D_n(q)$.

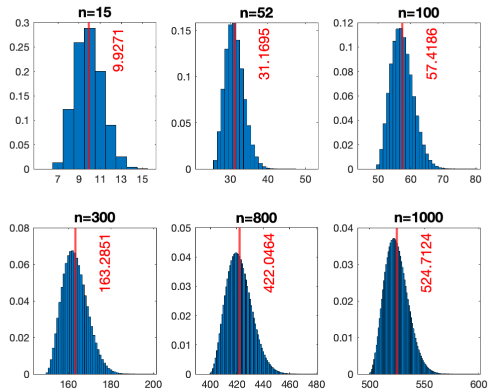

Figure 5: Probability histograms of the number of correct guesses when n varies. The red vertical line indicates the corresponding expected value $E[X_n]$.

The analysis of the asymptotic distribution of the number of correct guesses has been carried out by a group of Austrian Mathematicians, Kuba and Panholzer in 2025, [9].

Part VI

Card Guessing after k-riffle Shuffle, Feedback Version

During my talk on this topic about 4 years ago, I posted the question whether it is possible to extend this idea to k-riffle shuffle with $k \geq 2$. So far, nobody has done it yet.

Difficulty: Unlike 1-shuffle, there is no obvious optimal guessing strategy for k-shuffles, $k \geq 2$.

Observations:

- For k-shuffles, it is possible to get up to 2^k increasing subsequences. (As you've seen, the deck of cards after 1-shuffle has at most 2 increasing subsequences.
- Once the structure of the increasing subsequences is known (i.e. how many cards are left in each of the subsequences), the optimal guessing strategy is clear. That is guess the number in the longest (left) subsequences.

Solution:

Let's divide the guessing into two parts.

Part 1: When the structure of the increasing subsequences is not known yet.

Part 2: When the structure of the increasing subsequences is already known.

The plan could go as

1. We conjecture that the expected number of guesses in Part 1 is only a constant (for the deck of n cards).
2. As we know the optimal guessing strategy in Part 2, we can compute the number of correct guesses in this stage similar to what we did for 1-shuffle.
3. In total, we can approximate the expected number of correct guesses for k-shuffles to correct up to the constant term.

Unlike the 1-riffle shuffle case where the whole distribution of X (= number of correct guesses) is known, we might only get the approximate of $E[X]$ for the k-riffle shuffles, ($k \geq 2$). But this is how the research goes, we made one small step at a time.

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