Two games on arithmetic functions: Saliquant and Nontotient

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The rules of the game

We studied two of the many impartial, normal-play games on arithmetic funtions that were introduced by Ianucci and Larsson.
Their rules are as follows.

(a) SALIQUANT. Subtract a non-divisor: For $n \geq 1$, $\text{opt}(n) = \{n - k : 1 \leq k \leq n : k \nmid n\}$.

(b) NONTOTIENT. Subtract the number of relatively prime residues: For $n \geq 1$, $\text{opt}(n) = \{n - \phi(n)\}$, where ϕ is Euler's totient function.

In each case, the usual Sprague-Grundy theory applies.

$$\mathcal{SG}(n) = \text{mex}\{\mathcal{SG}(s), s \in opt(n)\}.$$

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For time, today we will only discuss SALIQUANT

Intro to SALIQUANT

SALIQUANT. Subtract a non-divisor: For $n \geq 1$,
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Larsson and Ianucci had already proved the following:

Lemma

- *If n is odd, then $\mathcal{SG}(n) = \frac{n-1}{2}$*
- *For all $n \geq 1$, $\mathcal{SG}(n) < \frac{n}{2}$*

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Proof.

If n is odd, then all smaller odd numbers are options, so by induction, the first part shows $\mathcal{SG}(n) \geq \frac{n-1}{2}$. By induction, the second part shows $\mathcal{SG}(n) \leq \frac{n-1}{2}$.
If n is even, then $n-1$ and $n-2$ are not options, so this case follows by induction. □

Even values

Our task, therefore, was to investigate the nim-values of even positions. The first few such values are:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28
$\mathcal{SG}(n)$	0	1	1	3	2	4	6	7	4	7	5	10	12	10

n	30	32	34	36	38	40	42	44	46	48	50	52	54
$\mathcal{SG}(n)$	13	15	8	13	9	17	17	16	11	22	22	19	25

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Any guesses for $\mathcal{SG}(2^b)$?

$\mathcal{SG}(2^b)$

Lemma

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Proof.

The numbers $3, 5, 7, \ldots, 2^b - 1$ are all non-divisors of 2^b. Thus $1, 3, 5, \ldots, 2^b - 3$ are all options, with corresponding nim-values $0, 1, 2, \ldots 2^{b-1} - 2$. Since $2^b - 2$ and $2^b - 1$ are not options, the lemma above implies that $2^{b-1} - 2$ is the largest nim-value of an option of 2^b □

A lower bound

Lemma

$\mathcal{SG}(n) \geq \frac{n-2}{4}$ *for all* n.

Proof.

This is already established for odd n.
Now let $n = 2^m k$, where k is odd and $m \geq 2$. Then $n-(k+2), n-(k+4), \ldots, 1$ are all options of n, with nim-values $\frac{1}{2}(n-k-3), \frac{1}{2}(n-k-5), \ldots, 0$, respectively. Thus

$$\mathcal{SG}(n) \geq \frac{1}{2}(n-k-1) \geq \frac{1}{2}\left(n - \frac{n}{4} - 1\right) > \frac{1}{2}\left(\frac{n}{2} - 1\right) = \frac{n-2}{4}$$

Next if $n = 2k$, where k is odd, then this talk is only 15 minutes □

A lower bound

Lemma

$\mathcal{SG}(n) \geq \frac{n-2}{4}$ *for all n.*

In fact, this lower bound is attained infinitely often:

Lemma

If p is prime and $p \equiv 5 \bmod 6$, *then* $\mathcal{SG}(2p) = \frac{p-1}{2} = \frac{2p-2}{4}$.

The key lemma

To give a flavor of later proofs, we show how to obtain what turned out to be an often used lemma.

Lemma

If $\mathcal{SG}(2n) = n - k$*, then* $2k - 1 \mid n$*.*

Proof.

Suppose $\mathcal{SG}(2n) = n - k$. Then $2n$ has no option of nim-value $n - k$. Since $\mathcal{SG}(2n - 2k + 1) = n - k$, it is not an option. In other words, $2k - 1 \mid 2n$ and hence $2k - 1 \mid n$. □

Fundamental Theorem of SALIQUANT

Recall:

Lemma

$\mathcal{SG}(2^b) = 2^{b-1} - 1$

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Which then extends similarly to:

Lemma

Let $b \geq 1$. *Then* $\mathcal{SG}((2a+1)2^b) = (2a+1)2^{b-1} - a - 1$ *for* $a = 0, 1, 2, 4$.

(and whose proofs fails for $a = 3$)

But it turns out that, using the key lemma from before, we can obtain a fully general formula....

Fundamental Theorem of SALIQUANT

Lemma

Let $b \geq 1$*. Then* $\mathcal{SG}((2a+1)2^b) = (2a+1)2^{b-1} - a - 1$ *for* $a = 0, 1, 2, 4$*.*

Theorem

For all $a \geq 0, b \geq 1$*,*

$$\begin{aligned}
&\mathcal{SG}\left((2a+1)2^b\right) \\
&= \frac{m}{2m+1}\left((2a+1)2^b - 1\right) + \frac{1}{2m+1}\left((2a+1)2^{b-1} - a - 1\right) \\
&= (2a+1)2^{b-1} - \frac{1}{2}\left(\frac{2a+1}{2m+1} + 1\right)
\end{aligned}$$

for some non negative integer m*. Thus*
$\mathcal{SG}\left((2a+1)2^b\right) = (2a+1)2^{b-1} - \frac{d+1}{2}$*, where* d *is a factor of* $2a+1$*.*

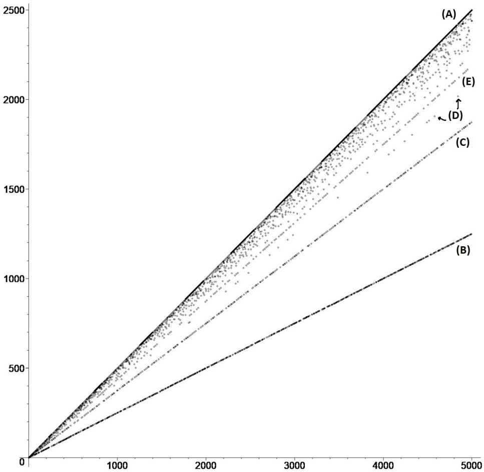
The distribution

Let $f(a,b,m) = \mathcal{SG}\left((2a+1)2^b\right)$ where m is chosen as in the fundamental theorem of Saliquant so that

$$\mathcal{SG}\left((2a+1)2^b\right) = (2a+1)2^{b-1} - \frac{1}{2}\left(\frac{2a+1}{2m+1} + 1\right).$$

m	$\mathcal{SG}(2(2a+1))$ $= f(a,1,m)$	density ($a \leq 5000$)	$\mathcal{SG}(4(2a+1))$ $= f(a,2,m)$	density ($a \leq 2000$)	$\mathcal{SG}(8(2a+1))$ $= f(a,3,m)$	density ($a \leq 2000$)
0	a **(B)**	0.532	$3a+1$ **(C)**	0.561	$7a+3$ **(E)**	0.540
1	$\frac{5a+1}{3}$ **(D)**	0.026	$\frac{11a+4}{3}$	0.056	$\frac{23a+10}{3}$	0.090
2	$\frac{9a+2}{5}$	0.037	$\frac{19a+7}{5}$	0.044	$\frac{39a+17}{5}$	0.050
3	$\frac{13a+3}{7}$	0.061	$\frac{27a+10}{7}$	0.049	$\frac{55a+24}{7}$	0.046
4	$\frac{17a+4}{9}$	0.022	$\frac{35a+13}{9}$	0.030	$\frac{71a+31}{9}$	0.015

The distribution



The first 5000 nim-values of SALIQUANT. The slopes of the labelled lines are **(A)** $\frac{1}{2}$, **(B)** $\frac{1}{4}$, **(C)** $\frac{3}{8}$, **(D)** $\frac{5}{12}$, **(E)** $\frac{7}{16}$

Thanks!

Maple code: **thotsaporn.com**

D. E. Iannucci, U. Larsson, "Game values of arithmetic functions". *Combinatorial Game Theory: A Special Collection in Honor of Elwyn Berlekamp, John H. Conway and Richard K. Guy*, edited by Richard J. Nowakowski, Bruce M. Landman, Florian Luca, Melvyn B. Nathanson, Jaroslav Nešetřil and Aaron Robertson, Berlin, Boston: De Gruyter, 2022, pp. 245-280.
`https://doi.org/10.1515/9783110755411-014`

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`https://doi.org/10.47443/dml.2023.154`