Generalizing OOOOOOB

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What is OOOOOOB?

The game One Or One Or One of Both, abbreviated OOOOOOB, is a classic puzzle:

> *There are two piles of tokens. On their turn, a player may take a token from one pile, take a token from the other pile, or take one token from each pile. The last player to move wins.*

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Proposition

In OOOOOOB, the $\mathcal{P}$-positions are precisely those where both piles are even.

We explored the $\mathcal{P}$-positions of three natural ways to generalize OOOOOOB to more than two piles.

Motivation

- The winning strategy for multi-pile NIM is a generalization of the winning strategy for the two-pile version.
- Similarly for DIVISOR NIM. See recent work of Takayuki Morisawa, or go back in time 2 days and watch his talk!

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So what about OOOOOOB? First, we must see how to generalize to more than 2 piles. There are 3 natural ways to do so:

A: A player selects any non-empty set of piles and takes one from each.

B: A player may remove any token *or* remove one token from all piles.

C: A player may remove any one token *or* select two piles and remove one token from each.

Warm-up and a curious lemma

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Lemma

If all piles are even, then it is a $\mathcal{P}$-position in all three versions.

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Can you see why?

Investigations

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We investigate along four lines:

I. Version B: few piles
II. Version B: few tokens per pile
III. Version C: few piles
IV. Version C: few tokens per pile

I. Results for Version B: few piles

B: A player may remove any token *or* remove one token from all piles.

Lemma

The $\mathcal{P}$-positions for 3 total piles are as follows, where e and o are used to denote any nonempty even or odd pile, respectively.

- e, e, e
- e, o, o

Lemma

The $\mathcal{P}$-positions for 4 total piles are as follows, where e denotes any even pile of size at least 2, and o denotes any odd pile of size at least 3.

- $1, 1, e, e$
- e, e, e, e
- e, o, o, o

I. Results for Version B: few piles

B: A player may remove any token *or* remove one token from all piles.

Lemma

The $\mathcal{P}$-positions for 5 or 6 total piles are as follows, where e denotes any even pile of size at least 2, and o denotes any odd pile of size at least 3.

- *Thirteen base cases which include 1s and/or 2s*
- e, e, e, e, e
- e, e, e, o, o
- e, o, o, o, o
- e, e, e, e, e, e
- e, e, e, e, o, o
- e, o, o, o, o, o

I. Results for Version B: few piles

B: A player may remove any token *or* remove one token from all piles.

Conjecture

Consider a position with some lower bound condition on the size of the piles. If there are k piles, then it is a $\mathcal{P}$-position precisely when:

- If k odd, then the number of odd piles is even
- If $k = 0 \bmod 4$, then the number of odd piles is $0, 2, 4, \ldots \frac{k}{2} - 2, \frac{k}{2} + 1 \ldots, k-5, k-3, k-1.$
- If $k = 2 \bmod 4$, then the number of odd piles is $0, 2, 4, \ldots \frac{k}{2} - 1, \frac{k}{2} + 2 \ldots, k-5, k-3, k-1.$

This has been verified for up to $k = 11$ piles, up to 10 tokens per pile.

II. Results for Version B: few tokens per pile

Lemma

Suppose there are a_i piles of size i for $1 \leq i \leq n$. Then the $\mathcal{P}$-positions for $n \leq 6$ are as follows.

- $(a_1, a_2) = (e, \geq 1)$
- $(a_1, a_2, a_3) = (o, o, 1), (e, o, \geq 2)$
- $(a_1, a_2, a_3, a_4) = 8$ *base cases,* $(e, o, e, \geq 4), (o, o, o, \geq 4)$
- $(a_1, a_2, a_3, a_4, a_5) = 26$ *base cases,*
  $(e, e, e, o, \geq 6), (e, e, o, o, \geq 6), (e, o, o, e, \geq 6), (e, o, e, e, \geq 6)$
- $(a_1, a_2, a_3, a_4, a_5, a_6) =$ *very many base cases;*
  $a_1,\ a_3 + a_5$ *even,* $a_6 \geq 8$*;*
  $a_1,\ a_2,\ a_3 + a_5$ *odd,* $a_6 \geq 8$

Conjecture

If $n \geq 3$, $a_n \geq 2n - 4$, this is a function of the parities of $a_1, \ldots, a_{n-1}$

III. Results for Version C: few piles

C: A player may remove any one token *or* select two piles and remove one token from each.

Lemma

If there are 3, 4, *or* 5 *piles, the* $\mathcal{P}$*-positions are as follows,* **where the piles are listed in nondecreasing order**.

- $\langle e, e, e \rangle$ $\quad$ $\langle o, o, o \rangle$
- $\langle e, e, e, e \rangle$ $\quad$ $\langle e, o, o, o \rangle$
- $\langle e, e, e, e, e \rangle$ $\quad$ $\langle e, e, o, o, o \rangle$
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So what about 6 piles?

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So what about 6 piles?
Version C for 6 piles is not determined by pure parity!
Instead, we gain more insight by observing few tokens per pile:

IV. Results for Version C: few tokens per pile

C: A player may remove any one token *or* select two piles and remove one token from each.

Lemma

Suppose that all piles have at most 3 tokens. In particular, there are a_1 piles of size 1, a_2 piles of size 2, and a_3 piles of size 3. Then the $\mathcal{P}$ positions are as follows:

- $a_1 = 0,\ a_3 = 0$
- $(a_1, a_2, a_3) = (0, 1, 3), (0, 2, 3), (1, 0, 2), (1, 0, 5), (2, 0, 1)$
- $a_1 + 2 \cdot a_2 = 0 \bmod 3$, *with the following exceptions:*
  - ⋆ $(a_1, a_2, a_3) = (0, 3k, e)$, *where* $e = 1, 2$
  - ⋆ $(a_1, a_2, a_3) = (1, 3k + 1, e)$, *where* $e = 0, 1$
  - ⋆ $(a_1, a_2, a_3) = (2, 3k + 2, e)$, *where* $e = 0$
  - $(a_1, a_2, a_3) = (0, 0, 4), (0, 0, 5), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 1, 5), (2, 2, 3), (3, 0, 1), (3, 0, 2), (3, 0, 5)$

IV. Results for Version C: few tokens per pile

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Lemma

Suppose there are a_1 piles of size 1*, a_2 piles of size* 2*, and a_3 piles of size* 3*. Then the $\mathcal{P}$ positions are as follows:*

- $a_1 = 0$, $a_3 = 0$ *(All piles are even)*
- $a_1 + 2 \cdot a_2 = 0 \bmod 3$ *(Total number of tokens is a multiple of* 3*)*

And some other exceptions

Note: if all piles have one token, then Version C $\cong$ $\textsc{subtract}(1, 2)$.

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Note: if all piles have one token, then Version C $\cong$ SUBTRACT$(1, 2)$.

Conjecture

Suppose we have a position of Version C with some lower bound condition. Then we have a $\mathcal{P}$-position precisely when the total number of tokens is a multiple of 3. (Verified up to 5 tokens per pile)

Some Theory

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The winning strategy for Version A generalizes to arbitrary sums of impartial games. That if we 'play Version A' on a sum of imparital game positions, then it is a $\mathcal{P}$-position iff all the summands are $\mathcal{P}$-positions.

Question

Is there some analogue for Version B or Version C?

Not for usual sums. For example, $\langle o, o, o \rangle$ is a $\mathcal{P}$-position for Version C, but if we 'play Version C' on the NIM-position $3, 1, 1$, we have the $\mathcal{P}$-option $1, 1, 1$. Maybe for another kind of sum?

Thanks!

Thanks to Alon, Paul, and the organizers of this conference!

Maple code: thotsaporn.com

Places oooooob is found in the wild:

S. Coskey, P. Ellis, J. Wood, *Five Fabulous Activities For Your Math Circle*, Natural Math, 2022.

D. Fomkin, S. Genkin, I. Itenberg *Mathematical Circles (Russian Experience)*, American Mathematical Society, 1996.

Motivation:

Takayuki Morisawa, *A really good talk about* DIVISOR NIM, here, two days ago.